A Common View on Strong, Uniform, and Other Notions of Equivalence in Answer-Set Programming*

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Abstract. Logic programming under the answer-set semantics nowadays deals with numerous different notions of equivalence between programs. This is due to the fact that equivalence for substitution (known as strong equivalence), which holds between programs P and Q iff P can faithfully be replaced by Q within any context R, is a different concept than ordinary equivalence between P and Q, which holds if P and Q have the same answer sets. Notions inbetween strong and ordinary equivalence have therefore been obtained by either restricting the syntactic structure of R or bounding the set of atoms allowed to occur in R (relativized equivalence). For the former approach, however, it turned out that any "reasonable" syntactic restriction to R either coincides with strong equivalence or collapses to uniform equivalence where R ranges over arbitrary sets of facts. In this paper, we propose a parameterization for equivalence notions which takes care of both such kinds of restrictions simultaneously by bounding, on the one hand, the atoms which are allowed to occur in the rule heads of R and, on the other hand, the atoms which are allowed to occur in the rule bodies of R. We introduce a semantical characterization including known ones as SE-models or UE-models as special cases. Moreover, we provide complexity bounds for the problem in question.

1 Introduction

Starting with the seminal paper on strong equivalence between logic programs by Lifschitz, Pearce, and Valverde [7], a new research direction in logic programming under the answer-set semantics has been established. This is due to fact that strong equivalence between programs P and Q, which holds iff P can faithfully be replaced by Q in any program, is a different concept than deciding whether P and Q have the same answer sets, i.e., (ordinary) equivalence between P and Q holds. Formally, P and Q are strongly equivalent iff, for each further so-called context program R, $P \cup R$ and $Q \cup R$ possess the same answer sets. That difference between strong and ordinary equivalence motivated investigations of equivalence notions inbetween (see, e.g., [4]). Basically this was done in two ways, viz. to bound the actually allowed context programs R by (i) restricting their syntax; or (ii) restricting their language. For Case (i), it turned out that any "reasonable" (i.e., where the restriction is defined rule-wise, for instance only allowing

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for Horn rules) attempt either coincides with strong equivalence itself, or reduces to uniform equivalence [2], which is to test whether, for each set F of facts, $P \cup F$ and $Q \cup F$ possess the same answer sets. Case (ii), where the atoms allowed to occur in R are given by an alphabet $\mathcal{A}$ yields in general different concepts for different $\mathcal{A}$ and thus is known as strong equivalence relative to $\mathcal{A}$ [12]. Finally a combination of both approaches leads to the concept of uniform equivalence relative to $\mathcal{A}$ [12].[1]

In this paper, we propose a fine-grained framework to define notions of equivalence where the aforementioned restrictions are simultaneously taken into account. This is accomplished by restricting, on the one hand, the atoms which are allowed to occur in the rule heads of the context programs and, on the other hand, the atoms which are allowed to occur in the rule bodies of the context programs. More formally, for given programs P, Q, and given sets $\mathcal{H}$, $\mathcal{B}$ of atoms, we want to decide whether the answer sets of $P \cup R$ and $Q \cup R$ coincide for each program R, where each rule in R has its head atoms from $\mathcal{H}$ and its body atoms from $\mathcal{B}$. We will show that this new notion includes all of the previously mentioned; for instance, setting $\mathcal{B} = \emptyset$, i.e., disallowing any atom to occur in bodies, will be shown to coincide with (relativized) uniform equivalence; while the parameterization $\mathcal{H} = \mathcal{B}$ amounts to (relativized) strong equivalence by definition.

The main contribution of the paper is to provide a general semantical characterization for the new equivalence notion. Moreover, we show that our characterization includes as special cases known concepts as SE-models [11] or UE-models [2]. Finally, we address the computational complexity of the introduced equivalence problems and propose a prototypical implementation.

2 Background

Throughout the paper we assume an arbitrary finite but fixed universe $\mathcal{U}$ of atoms. Subsets of $\mathcal{U}$ are either called interpretations or alphabets: We use the latter term to restrict the syntax of programs, while the former is used when talking about semantics. For an interpretation Y and an alphabet $\mathcal{A}$, we write $Y|_{\mathcal{A}}$ instead of $Y \cap \mathcal{A}$.

A propositional disjunctive logic program (or simply, a program) is a finite set of rules of form

$$a_1 \vee \cdots \vee a_l \leftarrow a_{l+1}, \ldots, a_m, not\, a_{m+1}, \ldots, not\, a_n, \quad (1)$$

$n > 0$, $n \geq m \geq l$, and where all a_i are propositional atoms in $\mathcal{U}$ and not denotes default negation; for $n = l = 1$, we usually identify the rule (1) with the atom a_1, and call it a *fact*. A rule of the form (1) is called a *constraint* if $l = 0$, *positive* if $m = n$ and *unary* if it is either a fact or of the form $a \leftarrow b$. A program is positive (resp., unary) iff all its rules are positive (resp., unary). If all atoms occurring in a program P are from a given alphabet $\mathcal{A} \subseteq \mathcal{U}$ of atoms, we say that P is a program *over* (alphabet) $\mathcal{A}$. The class of all logic programs over universe $\mathcal{U}$ is denoted by $\mathcal{C}_{\mathcal{U}}$.

For a rule r of form (1), we identify its head by $H(r) = \{a_1, \ldots, a_l\}$ and its body via $B^+(r) = \{a_{l+1}, \ldots, a_m\}$ and $B^-(r) = \{a_{m+1}, \ldots, a_n\}$. We shall write rules of

[1] A further approach is to additionally restrict the alphabet over which the answer sets of $P \cup R$ and $Q \cup R$ compared. This kind of *projection* was investigated in [5, 8, 10], but we do not consider it in this work.

form (1) also as $H(r) \leftarrow B^+(r), not\, B^-(r)$. Moreover, we also use $B(r) = B^+(r) \cup B^-(r)$. Finally, for a program P, $\alpha(P) = \bigcup_{r \in P} \alpha(r)$, for $\alpha \in \{H, B, B^+, B^-\}$.

The relation $Y \models P$ between an interpretation Y and a program P is defined as usual, i.e., $Y \models P$ holds if for each $r \in P$, $Y \models r$. The latter holds iff $H(r) \cap Y \neq \emptyset$, whenever jointly $B^+(r) \subseteq Y$ and $B^-(r) \cap Y = \emptyset$ hold. If $Y \models P$ holds, Y is called a model of P. Following Gelfond and Lifschitz [6], an interpretation Y, is an *answer set* of a program P iff it is a minimal (wrt set inclusion) model of the *reduct* $P^Y = \{H(r) \leftarrow B^+(r) \mid Y \cap B^-(r) = \emptyset\}$. The set of all answer sets of a program P is denoted by $\mathcal{AS}(P)$.

Finally, we briefly review some prominent notions of equivalence [7, 2, 12, 4], which have been studied under the answer-set semantics: For a given alphabet $\mathcal{A} \subseteq \mathcal{U}$, we call programs $P, Q \in \mathcal{C}_\mathcal{U}$, *strongly equivalent relative to* $\mathcal{A}$, iff, for any program R over $\mathcal{A}$, it holds that $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$. P, Q are *uniformly equivalent relative to* $\mathcal{A}$, iff, for any set $F \subseteq \mathcal{A}$ of facts, $\mathcal{AS}(P \cup F) = \mathcal{AS}(Q \cup F)$. If, $\mathcal{A} = \mathcal{U}$, strong (resp., uniform) equivalence relative to $\mathcal{A}$ collapses to (unrelativized) strong (resp., uniform) equivalence [7, 2]; if $\mathcal{A} = \emptyset$, we obtain *ordinary equivalence*, i.e., $\mathcal{AS}(P) = \mathcal{AS}(Q)$.

In case of strong equivalence (also in the relativized case), it was shown that the syntactic class of *counterexamples*, i.e., programs R, such that $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$, can always be restricted to the class of unary programs. Hence, the next result comes by mere surprise, but provides insight with respect to the alphabets in the rules' heads and bodies.

Lemma 1. *Let P, Q, $R \in \mathcal{C}_\mathcal{U}$ be programs, and Y be an interpretation, such that $Y \in \mathcal{AS}(P \cup R)$ and $Y \notin \mathcal{AS}(Q \cup R)$. Then there exists a program R', such that R' is positive, $H(R') \subseteq H(R)$, $B(R') \subseteq B(R)$, $Y \in \mathcal{AS}(P \cup R')$, and $Y \notin \mathcal{AS}(Q \cup R')$.*

The result can be checked by using $R' = R^Y$.

As we will see later, Lemma 1 can even be strengthened to unary programs. However, already the present result shows that whenever a counterexample R for an equivalence problem exists, then we can find a simpler (positive) one, which is given over the same alphabets in the heads, and respectively, bodies.

3 The General Framework

Lemma 1 suggests to study equivalence problems along a parameterization via two alphabets. To this end, we first introduce classes of programs as follows.

Definition 1. *For any alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$, the class $\mathcal{C}_{\langle \mathcal{H}, \mathcal{B} \rangle}$ of programs is defined as $\{P \in \mathcal{C}_\mathcal{U} \mid H(P) \subseteq \mathcal{H}, B(P) \subseteq \mathcal{B}\}$.*

With this concept of program classes at hand, we now define equivalence notions which are more fine-grained than the ones previously introduced.

Definition 2. *Let $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$ be alphabets, and $P, Q \in \mathcal{C}_\mathcal{U}$ be programs. The $\langle \mathcal{H}, \mathcal{B} \rangle$-*equivalence problem *between P and Q, in symbols $P \equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$, is to decide whether, for each $R \in \mathcal{C}_{\langle \mathcal{H}, \mathcal{B} \rangle}$, $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$. If $P \equiv_{\langle \mathcal{H}, \mathcal{B} \rangle} Q$ holds, we say that P and Q are $\langle \mathcal{H}, \mathcal{B} \rangle$-*equivalent.

The class $\mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$ is also called the *context* of an $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence problem, and a program $R \in \mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$, where $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$ holds, is called a *counterexample* to the $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence problem between P and Q.

Example 1. Consider $P = \{a \vee b \leftarrow;\ a \leftarrow b\}$ and $Q = \{a \leftarrow not\, b;\ b \leftarrow not\, a;\ a \leftarrow b\}$. It is known that these programs are not strongly equivalent, since adding any R which closes the cycle between a and b yields $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$. In particular, for $R = \{b \leftarrow a\}$, we get $\mathcal{AS}(P \cup R) = \{\{a,b\}\}$, while $\mathcal{AS}(Q \cup R) = \emptyset$. However, P and Q are uniformly equivalent. In our setting, we are able to "approximate" equivalence notions which hold between P and Q. It can be shown that, for instance, $P \equiv_{\langle\{a,b\},\{b\}\rangle} Q$ or $P \equiv_{\langle\{a\},\{a,b\}\rangle} Q$ holds (basically since $b \leftarrow a$ does not occur in any program in $\mathcal{C}_{\langle\{a,b\},\{b\}\rangle}$, or $\mathcal{C}_{\langle\{a\},\{a,b\}\rangle}$). But $P \equiv_{\langle\{b\},\{a,b\}\rangle} Q$ and likewise $P \equiv_{\langle\{a,b\},\{a\}\rangle} Q$ do not hold, since $\{b \leftarrow a\}$ is contained in the context $\mathcal{C}_{\langle\{b\},\{a,b\}\rangle}$, resp., $\mathcal{C}_{\langle\{a,b\},\{a\}\rangle}$. ◇

Observe that the concept of $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence captures other equivalence notions as follows: $\langle\mathcal{A},\mathcal{A}\rangle$-equivalence coincides with strong equivalence relative to $\mathcal{A}$; and, in particular, $\langle\mathcal{U},\mathcal{U}\rangle$-equivalence amounts to strong equivalence. Later we will see that $\langle\mathcal{A},\emptyset\rangle$-equivalence coincides with uniform equivalence relative to $\mathcal{A}$; and, in particular, $\langle\mathcal{U},\emptyset\rangle$-equivalence amounts to uniform equivalence. Note that the relation to uniform equivalence is not immediate since $\langle\mathcal{A},\emptyset\rangle$-equivalence deals with sets of *disjunctive* facts, i.e., rules of the form $a_1 \vee \cdots \vee a_l \leftarrow$, rather than sets of (simple) facts $a \leftarrow$.

The following result shows some general properties for $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence.

Proposition 1. *Let $\mathcal{H},\mathcal{B} \subseteq \mathcal{U}$ and $P,Q \in \mathcal{C}_\mathcal{U}$, such that $P \equiv_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ holds. Then, also $(P \cup R) \equiv_{\langle\mathcal{H}',\mathcal{B}'\rangle} (Q \cup R)$ holds, for each $R \in \mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$, $\mathcal{H}' \subseteq \mathcal{H}$, and $\mathcal{B}' \subseteq \mathcal{B}$.*

A central aspect in equivalence checking is the quest for semantical characterizations assigned to a *single* program. The following formal approach captures this aim.

Definition 3. *A* semantical characterization *for an $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence problem is a function $\sigma_{\langle\mathcal{H},\mathcal{B}\rangle} : \mathcal{C}_\mathcal{U} \rightarrow 2^{2^\mathcal{U} \times 2^\mathcal{U}}$, such that, for any $P,Q \in \mathcal{C}_\mathcal{U}$, $P \equiv_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ holds iff $\sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P) = \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$.*

We will review known characterizations for special cases (as, for instance, SE-models [11] and UE-models [2]) later. Finally, we also introduce containment problems.

Definition 4. *Let $\mathcal{H},\mathcal{B} \subseteq \mathcal{U}$ be alphabets, and $P,Q \in \mathcal{C}_\mathcal{U}$ be programs. The $\langle\mathcal{H},\mathcal{B}\rangle$-*containment problem *for P in Q, in symbols $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$, is to decide whether, for each $R \in \mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$, $\mathcal{AS}(P \cup R) \subseteq \mathcal{AS}(Q \cup R)$. A counterexample to $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$, is any program $R \in \mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$, such that $\mathcal{AS}(P \cup R) \not\subseteq \mathcal{AS}(Q \cup R)$.*

Proposition 2. *$P \equiv_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ holds iff $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ and $Q \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} P$ jointly hold.*

4 Characterizations for $\langle\mathcal{H},\mathcal{B}\rangle$-Equivalence

Towards the semantical characterization for $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence problems, we first introduce the notion of a witness, which is assigned to $\langle\mathcal{H},\mathcal{B}\rangle$-containment problems taking both compared programs into account. Afterwards, we will derive the desired semantical characterization of $\langle\mathcal{H},\mathcal{B}\rangle$-models which are assigned to single programs and satisfy the conditions in Definition 3.

To start with, we introduce the following partial order on interpretations and state a technical lemma.

Definition 5. *Given alphabets $\mathcal{H},\mathcal{B}\subseteq\mathcal{U}$, we define the relation $\preceq^{\mathcal{B}}_{\mathcal{H}}\subseteq\mathcal{U}\times\mathcal{U}$ between interpretations as follows: $V\preceq^{\mathcal{B}}_{\mathcal{H}} Z$ iff $V|_{\mathcal{H}}\subseteq Z|_{\mathcal{H}}$ and $Z|_{\mathcal{B}}\subseteq V|_{\mathcal{B}}$.*

Observe that if $V \preceq^{\mathcal{B}}_{\mathcal{H}} Z$ holds, then either $V|_{\mathcal{H}\cup B} = Z|_{\mathcal{H}\cup B}$, or one of $V|_{\mathcal{H}} \subset Z|_{\mathcal{H}}$, $Z|_{\mathcal{B}}\subset V|_{\mathcal{B}}$ holds. We write $V\prec^{\mathcal{B}}_{\mathcal{H}} Z$, in case $V\preceq^{\mathcal{B}}_{\mathcal{H}} Z$ and $V|_{\mathcal{H}\cup B}\neq Z|_{\mathcal{H}\cup B}$.

Lemma 2. *Let $\mathcal{H},\mathcal{B}\subseteq\mathcal{U}$ be alphabets, P a positive program with $H(P)\subseteq\mathcal{H}$, $B(P)\subseteq\mathcal{B}$, and $Z,V\subseteq\mathcal{U}$ interpretations. Then, $V\models P$ and $V\preceq^{\mathcal{B}}_{\mathcal{H}} Z$ imply $Z\models P$.*

Proof. Towards a contradiction, suppose $V\models P$, $V|_{\mathcal{H}}\subseteq Z|_{\mathcal{H}}$, $Z|_{\mathcal{B}}\subseteq V|_{\mathcal{B}}$, as well as $Z\not\models P$ hold. If $Z\not\models P$, then there exists a rule $r\in P$, such that $B^+(r)\subseteq Z$ and $Z\cap H(r)=\emptyset$. Since $H(r)\subseteq\mathcal{H}$, we get from $V|_{\mathcal{H}}\subseteq Z|_{\mathcal{H}}$, that $V\cap H(r)=\emptyset$. Moreover, since $B^+(r)\subseteq\mathcal{B}$, we have $B^+(r)\subseteq Z|_{\mathcal{B}}\subseteq V|_{\mathcal{B}}$, and thus $B^+(r)\subseteq V$. Hence $V\not\models r$ which yields $V\not\models P$. Contradiction. □

4.1 Witnesses for Containment Problems

Definition 6. *A* witness *for (violating) a containment problem $P\subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ is a pair of interpretations (X,Y) with $X\subseteq Y\subseteq\mathcal{U}$, such that*

(i) $Y\models P$ and for each $Y'\subset Y$, $Y'\models P^Y$ implies $Y'|_{\mathcal{H}}\subset Y|_{\mathcal{H}}$;
(ii) if $Y\models Q$ then $X\subset Y$, $X\models Q^Y$, and for each X' with $X\preceq^{\mathcal{B}}_{\mathcal{H}} X'\subset Y$, $X'\not\models P^Y$.

The aim of a witness (X,Y) for (violating) $P\subseteq_{\langle\mathcal{H},B\rangle} Q$ is, roughly speaking, as follows: Set X is used to characterize a counterexample R, such that set Y behaves as a witnessing answer set, i.e., $Y\in\mathcal{AS}(P\cup R)$ and $Y\notin\mathcal{AS}(Q\cup R)$. Property (i) ensures that Y can become such an answer set of an extended P. To this end, it is not only necessary that $Y\models P$. It also has to be guaranteed that no $Y'\subset Y$, with $Y'|_{\mathcal{H}}=Y|_{\mathcal{H}}$ satisfies $Y'\models P^Y$, otherwise Y can never become an answer of $P\cup R$, no matter which $R\in\mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$ is added to P. Property (ii) ensures that the program R is obtained from X in such a way, that Y does not become an answer set of $Q\cup R$, but Y still can become an answer set of $P\cup R$. We can focus on a positive program R (cf. Lemma 1), and R can be constructed in such a way, that it rules out all possible models $X'\subset Y$ of P^Y, as long as $X\not\preceq^{\mathcal{B}}_{\mathcal{H}} X'$ holds. The latter is due to the fact that each positive $R\in\mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$ suitably applies here to Lemma 2.

Example 2. We already have mentioned that $P=\{a\vee b\leftarrow;\ a\leftarrow b\}$ and $Q=\{a\leftarrow \mathit{not}\,b;\ b\leftarrow \mathit{not}\,a;\ a\leftarrow b\}$ are not $\langle\mathcal{H},\mathcal{B}\rangle$-equivalent for $\mathcal{H}=\{b\}$ and $\mathcal{B}=\{a,b\}$. We show that there exists a witness for $P\subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$. First, let us compute the programs' models (over $\{a,b\}$) as well as the models of their reducts. Observe that P and Q have the same models $Y_1=\{a,b\}$ and $Y_2=\{a\}$. For the positive program P we are done, since all reducts coincide with P, and thus possess the same models. For Q, however, observe that $Q^{Y_1} = \{a \leftarrow b\}$ has models $\emptyset$, $\{a\}$, and $\{a,b\}$, while $Q^{Y_2} = \{a;\ a \leftarrow b\}$ has models $\{a\}$ and $\{a,b\}$. We show that for $X = \emptyset$, (X, Y_1) is a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$. Clearly, Condition (i) from Definition 6 holds, since $Y_1 \models P$ and $Y_2|_{\mathcal{H}} \subset Y_1|_{\mathcal{H}}$. Concerning Condition (ii), we have $Y_1 \models Q$, $X \subset Y_1$, and $X \models Q^{Y_1}$. The only X' (over $\{a,b\}$) such that $X \preceq^{\mathcal{B}}_{\mathcal{H}} X'$ holds is X itself, since $\mathcal{B} = \{a,b\}$ and thus $X' \subseteq X$ has to be satisfied. It thus remains to check $X \not\models P^{Y_1}$, which is the case. Hence, $(\emptyset, \{a,b\})$ is a witness for $P \subseteq_{\langle\{b\},\{a,b\}\rangle} Q$. By similar arguments (in particular, since also $\{b\} \not\models P^{Y_1}$), $(\emptyset, \{a,b\})$ is a witness also for $P \subseteq_{\langle\{a,b\},\{a\}\rangle} Q$. $\diamond$

We now formally proof that the existence of witnesses for a containment problem $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ in fact shows that $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ does not hold. As a by-product we obtain that there are always counterexamples to $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ of a simple syntactic form.

Lemma 3. *The following propositions are equivalent for any $P, Q \in \mathcal{C}_{\mathcal{U}}$, $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$:*

(1) $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ does not hold;
(2) there exists a unary program $R \in \mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$, such that $\mathcal{AS}(P \cup R) \not\subseteq \mathcal{AS}(Q \cup R)$;
(3) there exists a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$.

Proof. We show that (1) implies (3) and (3) implies (2). (2) implies (1) obviously holds by definition of $\langle\mathcal{H},\mathcal{B}\rangle$-containment problems.

(1) implies (3): If $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ does not hold, there exists a program R, and an interpretation Y, such that $Y \in \mathcal{AS}(P \cup R)$ and $Y \notin \mathcal{AS}(Q \cup R)$. By Lemma 1, we can wlog assume that R is positive. Moreover, we know $H(R) \subseteq \mathcal{H}$ and $B(R) \subseteq \mathcal{B}$. Starting from $Y \in \mathcal{AS}(P \cup R)$, we first show that Property (i) from Definition 6 holds. We have $Y \models P \cup R$, and thus, $Y \models P$ as well as $Y \models R$ holds. It remains to show that for each $Y' \subset Y$, $Y' \models P^Y$ implies $Y'|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$. Towards a contradiction, now suppose there exists an $Y' \subset Y$ such that $Y' \models P^Y$ and $Y'|_{\mathcal{H}} \not\subset Y|_{\mathcal{H}}$. Since $Y' \subset Y$, we have $Y'|_{\mathcal{H}} = Y|_{\mathcal{H}}$, and thus, $Y|_{\mathcal{H}} \subseteq Y'|_{\mathcal{H}}$. Moreover, $Y'|_{\mathcal{B}} \subseteq Y|_{\mathcal{B}}$ holds, and we get $Y \preceq^{\mathcal{B}}_{\mathcal{H}} Y'$. By $Y \models R$ and Lemma 2 this yields $Y' \models R$. But then $Y' \models (P^Y \cup R) = (P \cup R)^Y$, a contradiction to $Y \in \mathcal{AS}(P \cup R)$.

It remains to establish Property (ii) in Definition 6. From $Y \notin \mathcal{AS}(Q \cup R)$, we either get $Y \not\models Q \cup R$ or existence of an X such that $X \models (Q \cup R)^Y = (Q^Y \cup R)$. We already know that $Y \models R$. Hence, in the former case, i.e., $Y \not\models Q \cup R$, we get $Y \not\models Q$. Then, for any $X \subseteq Y$, (X, Y) is a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$, and we are done. For the remaining case, where $X \models Q^Y$ and $X \models R$, we suppose towards a contradiction, that there exists an $X' \subset Y$, such that $X' \models P^Y$ and $X \preceq^{\mathcal{B}}_{\mathcal{H}} X'$ hold. The latter together with $X \models R$ yields $X' \models R$, following Lemma 2. Together with $X' \models P^Y$, we thus get $X' \models (P^Y \cup R) = (P \cup R)^Y$. Since $X' \subset Y$ this is in contradiction to $Y \in \mathcal{AS}(P \cup R)$. Thus (X, Y) is a witness for for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$.

(3) implies (2): Let (X, Y) be a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$. We use the unary program

$$R = X|_{\mathcal{H}} \cup \{a \leftarrow b \mid a \in (Y \setminus X)|_{\mathcal{H}}, b \in (Y \setminus X)|_{\mathcal{B}}\}$$

and show $Y \in \mathcal{AS}(P \cup R) \setminus AS(Q \cup R)$. We first show $Y \in \mathcal{AS}(P \cup R)$. Since (X, Y) is a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$, we know $Y \models P$. $Y \models R$ is easily checked and thus $Y \models P \cup R$. It remains to show that no $Z \subset Y$ satisfies $Z \models (P \cup R)^Y = P^Y \cup R$. Towards a contradiction suppose such a Z exists. Hence, $Z \models P^Y$ and $Z \models R$. By $Z \models R$, $X|_{\mathcal{H}} \subseteq Z|_{\mathcal{H}}$ has to hold. Since (X, Y) is a witness for $P \subseteq_{\langle \mathcal{H},\mathcal{B} \rangle} Q$, $Z|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$ holds, otherwise Property (i) in Definition 6 is violated. Hence, $X|_{\mathcal{H}} \subseteq Z|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$ holds. We have $X \subset Y$ and, moreover, get $Z|_{\mathcal{B}} \not\subseteq X|_{\mathcal{B}}$ from Property (ii) in Definition 6, since $Z \models P^Y$ and $X|_{\mathcal{H}} \subseteq Z|_{\mathcal{H}}$ already hold. Now, $Z|_{\mathcal{B}} \subseteq Y|_{\mathcal{B}}$ by assumption, hence there exists an atom $b \in (Y \setminus X)|_{\mathcal{B}}$ contained in Z. We already know that $X|_{\mathcal{H}} \subseteq Z|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$ has to hold. Hence, there exists at least one $a \in (Y \setminus X)|_{\mathcal{H}}$, not contained in Z. But then, we derive that $Z \not\models \{a \leftarrow b\}$. Since $a \leftarrow b \in R$, this is a contradiction to $Z \models R$.

It remains to show $Y \notin \mathcal{AS}(Q \cup R)$. If $Y \not\models Q$, we are done. So let $Y \models Q$. Since (X, Y) is a witness for $P \subseteq_{\langle \mathcal{H},\mathcal{B} \rangle} Q$, we get $X \models Q^Y$ and $X \subset Y$. It is easy to see that $X \models R$ holds. Thus $X \models (Q^Y \cup R) = (Q \cup R)^Y$; $Y \notin \mathcal{AS}(Q \cup R)$ follows. □

As an immediate consequence of Lemma 3 and Proposition 2, we get that $\langle \mathcal{H},\mathcal{B} \rangle$-equivalence problems which do not hold always possess simple counterexamples. As a special case we obtain the already mentioned fact that $\langle \mathcal{H},\emptyset \rangle$-equivalence amounts to uniform equivalence relative to $\mathcal{H}$.

Corollary 1. *For any $\mathcal{H},\mathcal{B} \in \mathcal{U}$ and programs $P, Q \in \mathcal{C}_{\mathcal{U}}$, $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ does not hold iff there exists a unary program $R \in \mathcal{C}_{\langle \mathcal{H},\mathcal{B} \rangle}$, such that $\mathcal{AS}(P \cup R) \neq \mathcal{AS}(Q \cup R)$; if $\mathcal{B} = \emptyset$, then $P \equiv_{\langle \mathcal{H},\mathcal{B} \rangle} Q$ does not hold iff there exists a set $F \subseteq \mathcal{H}$ of facts, such that $\mathcal{AS}(P \cup F) \neq \mathcal{AS}(Q \cup F)$.*

4.2 Introducing $\langle \mathcal{H},\mathcal{B} \rangle$-models

Next, we present the desired semantical characterization for $\langle \mathcal{H},\mathcal{B} \rangle$-equivalence, which we call $\langle \mathcal{H},\mathcal{B} \rangle$-models. First, we introduce two further properties.

Definition 7. *Given $\mathcal{H} \subseteq \mathcal{U}$, an interpretation Y is an $\mathcal{H}$-*total model *for P iff $Y \models P$ and for all $Y' \subset Y$, $Y' \models P^Y$ implies $Y'|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$.*

Definition 8. *Given $\mathcal{H},\mathcal{B} \subseteq \mathcal{U}$, a pair (X, Y) of interpretations is called $\preceq^{\mathcal{B}}_{\mathcal{H}}$-maximal for P iff $X \models P^Y$ and, for each X' with $X \prec^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, $X' \not\models P^Y$.*

Observe that Y being an $\mathcal{H}$-total model for P matches Property (i) from Definition 6 and follows the same intuition. Being $\preceq^{\mathcal{B}}_{\mathcal{H}}$-maximal refers to being maximal (wrt subset inclusion) in the atoms from $\mathcal{H}$ and simultaneously minimal in the atoms from $\mathcal{B}$.

Definition 9. *Given $\mathcal{H},\mathcal{B} \subseteq \mathcal{U}$, and interpretations $X \subseteq Y \subseteq \mathcal{U}$, a pair (X, Y) is an $\langle \mathcal{H},\mathcal{B} \rangle$-*model *of a program $P \in \mathcal{C}_{\mathcal{U}}$ iff Y is an $\mathcal{H}$-total model for P and, if $X \subset Y$, there exists an $X' \subset Y$ with $X'|_{\mathcal{H} \cup \mathcal{B}} = X$, such that (X', Y) is $\preceq^{\mathcal{B}}_{\mathcal{H}}$ maximal for P. The set of all $\langle \mathcal{H},\mathcal{B} \rangle$-models of a program P is denoted by $\sigma_{\langle \mathcal{H},\mathcal{B} \rangle}(P)$.*

Moreover, we call a pair (X, Y) *total* if $X = Y$, otherwise it is called *non-total*. Observe that each non-total $\langle \mathcal{H},\mathcal{B} \rangle$-model (X, Y) satisfies $X \subseteq Y|_{\mathcal{H} \cup \mathcal{B}}$ and $X|_{\mathcal{H}} \subset Y|_{\mathcal{H}}$.

Example 3. In Example 1, we already mentioned that $P = \{a \vee b \leftarrow;\ a \leftarrow b\}$ and $Q = \{a \leftarrow not\, b;\ b \leftarrow not\, a;\ a \leftarrow b\}$ are $\langle \{a, b\}, \{b\} \rangle$-equivalent. Hence, fix $\mathcal{H} = \{a, b\}$, $\mathcal{B} = \{b\}$, and let us compute the $\langle \mathcal{H},\mathcal{B} \rangle$-models of P, and resp., Q. In Example 2 we already have obtained the models of these programs as well as their reducts. There, we have seen that $Y_1 = \{a,b\}$ and $Y_2 = \{a\}$ are the models of both P and Q. Since $\mathcal{H} = \{a,b\}$, both are $\mathcal{H}$-total models for P and Q. So, (Y_1,Y_1) and (Y_2,Y_2) are the total $\langle\mathcal{H},\mathcal{B}\rangle$-models of both programs. It remains to check whether the non-total $\langle\mathcal{H},\mathcal{B}\rangle$-models of P and Q coincide. First observe that (Y_2,Y_1) is $\langle\mathcal{H},\mathcal{B}\rangle$-model of both P and Q, as well. The interesting candidate is $(\emptyset,Y_1)$ since $\emptyset$ is model of Q^{Y_1} but not of P^{Y_1}. Hence, $(\emptyset,Y_1)$ cannot be $\langle\mathcal{H},\mathcal{B}\rangle$-model of P. But $(\emptyset,Y_1)$ is also not $\langle\mathcal{H},\mathcal{B}\rangle$-model of Q, since there exists an interpretation X' satisfying $\emptyset \prec^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, which is model of Q^{Y_1}, viz. $X' = \{a\}$. In fact, $\emptyset|_{\mathcal{H}} \subset X'|_{\mathcal{H}}$ and $\emptyset|_{\mathcal{B}} = X'|_{\mathcal{B}}$ hold.

For $\mathcal{H} = \{a\}$ and $\mathcal{B} = \{a,b\}$, one can show that (Y_2,Y_2) is the only $\langle\mathcal{H},\mathcal{B}\rangle$-model (over $\{a,b\}$) of P as well as of Q, since Y_1 is no $\mathcal{H}$-total model in this setting. $\diamond$

Before stating our main theorem, we require one further lemma.

Lemma 4. *Let $P,Q \in \mathcal{C}_{\mathcal{U}}$, $\mathcal{H},\mathcal{B} \subseteq \mathcal{U}$, and Y be an interpretation. Then, $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P) \setminus \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$ iff there is a witness (X,Y) to $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ with $X|_{\mathcal{H}} = Y|_{\mathcal{H}}$.*

Proof. For the only-if direction, we directly obtain from $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P)$, that Property (i) in Definition 6 holds. To show the remaining Property (ii), observe that from $(Y,Y) \notin \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$, we either have $Y \not\models Q$ or existence of some $Y' \subset Y$, such that $Y' \models Q^Y$ and $Y'|_{\mathcal{H}} = Y|_{\mathcal{H}}$. In the former case, we are already done, and get that any (X,Y) with $X \subseteq Y$ is a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$, in particular for $X|_{\mathcal{H}} = Y|_{\mathcal{H}}$. It remains to show that, in case $Y \models Q$, and for some $Y' \subset Y$ with $Y'|_{\mathcal{H}} = Y|_{\mathcal{H}}$, $Y' \models Q^Y$, each X' with $Y' \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$ satisfies $X' \not\models P^Y$. By definition, this would make (Y',Y) a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$. Towards a contradiction, suppose such an X' exists. But then, from $Y' \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$ and $Y'|_{\mathcal{H}} = Y|_{\mathcal{H}}$, we get $Y'|_{\mathcal{H}} = X'|_{\mathcal{H}} = Y|_{\mathcal{H}}$. Thus, Y cannot be an $\mathcal{H}$-total model of P; a contradiction to $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P)$.

For the if-direction, let (X,Y) be a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$. Property (i) in Definition 6 yields $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P)$. It remains to show $(Y,Y) \notin \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$. Now, (X,Y) being a witness implies that either $Y \not\models Q$ or $X \models Q^Y$, where $X|_{\mathcal{H}} = Y|_{\mathcal{H}}$ and $X \subset Y$ hold. Both cases prevent (Y,Y) from being $\langle\mathcal{H},\mathcal{B}\rangle$-model of Q. $\square$

Theorem 1. *For any programs $P,Q \in \mathcal{C}_{\mathcal{U}}$ and alphabets $\mathcal{H},\mathcal{B} \subseteq \mathcal{U}$, $P \equiv_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ holds iff $\sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P) = \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$.*

Proof. If-direction: Suppose that either $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ or $Q \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} P$ does not hold. Let us wlog assume $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ does not hold (the other case is symmetric). By Lemma 3, then a witness (X,Y) to $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$ exists. By Property (i) in Definition 6, we immediately get $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P)$. In case $(Y,Y) \notin \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$ we are already done. So suppose $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$. Hence, we can assume $Y \models Q$, and by Lemma 4, $X|_{\mathcal{H}} \neq Y|_{\mathcal{H}}$. Since (X,Y) is a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$, we get that $X \models Q^Y$ holds, and for each X' with $X \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, $X' \not\models P^Y$. Consider now an arbitrary pair (Z,Y) of interpretations with $Z \subset Y$ which is $\preceq^{\mathcal{B}}_{\mathcal{H}}$-maximal for Q. Then $X \preceq^{\mathcal{B}}_{\mathcal{H}} Z$ has to hold and since $(Y,Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$, Y is an $\mathcal{H}$-total model of Q, and we obtain $(Z|_{\mathcal{H}\cup\mathcal{B}},Y) \in \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(Q)$. On that other hand, $(Z|_{\mathcal{H}\cup\mathcal{B}},Y) \notin \sigma_{\langle\mathcal{H},\mathcal{B}\rangle}(P)$ holds. This is a consequence of the observation that for each X' with $X \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, $X' \not\models P^Y$, (since (X,Y) is a witness for $P \subseteq_{\langle\mathcal{H},\mathcal{B}\rangle} Q$), and by the fact that $X \preceq^{\mathcal{B}}_{\mathcal{H}} Z$.

Only-if direction: Wlog assume $(X, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(P) \setminus \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$; again, the other case is symmetric. From $(X, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(P)$, $(Y, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(P)$ follows by Definition 9. Hence, if $(Y, Y) \notin \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$, we are done, since we know from Lemma 4 that then, there exists a witness for $P \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} Q$ and we get by Lemma 3, that $P \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} Q$ does not hold. Consequently, $P \equiv_{\langle \mathcal{H},\mathcal{B}\rangle} Q$ cannot hold as well. Thus, let $X \subset Y$, and $(Y, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$. We distinguish between two cases: First suppose there exists an X' with $X \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, such that $(X', Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$. Since $(X, Y) \notin \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$, by definition of $\langle \mathcal{H},\mathcal{B}\rangle$-models, $X \prec^{\mathcal{B}}_{\mathcal{H}} X'$ has to hold, and there exists a $Z \subset Y$ with $Z|_{\mathcal{H}\cup\mathcal{B}} = X'$, such that $Z \models Q^Y$. We show that (Z, Y) is a witness for $P \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} Q$. Since $(Y, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(P)$, Property (i) in Definition 6 holds. We know $Z \models Q^Y$, and since $(X, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(P)$, we get by definition of $\langle \mathcal{H},\mathcal{B}\rangle$-models, that, for each X'' with $X \prec^{\mathcal{B}}_{\mathcal{H}} X'' \subset Y$, $X'' \not\models P^Y$. Now since $X \prec^{\mathcal{B}}_{\mathcal{H}} Z$, Property (ii) in Definition 6 holds for Z (instead of X) as well. This shows that (Z, Y) is a witness for $P \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} Q$. So suppose, for each X' with $X \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$, $(X', Y) \notin \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$ holds. We have $(X, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(P)$, thus there exists a $Z \subset Y$, with $Z|_{\mathcal{H}\cup\mathcal{B}} = X$, such that $Z \models P^Y$. We show that (Z, Y) is a witness for the reverse problem, $Q \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} P$. From $(Y, Y) \in \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$, we get that Property (i) in Definition 6 is satisfied for Q and Y. Moreover, we have $Z \models P^Y$. It remains to show that, for each X'' with $X \preceq^{\mathcal{B}}_{\mathcal{H}} X'' \subset Y$, $X'' \not\models Q^Y$. This holds by assumption, i.e., $(X', Y) \notin \sigma_{\langle \mathcal{H},\mathcal{B}\rangle}(Q)$, for each X' with $X \preceq^{\mathcal{B}}_{\mathcal{H}} X' \subset Y$. Hence, both cases yield a witness, either for $P \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} Q$ or $Q \subseteq_{\langle \mathcal{H},\mathcal{B}\rangle} P$. By Lemma 3 and Proposition 2, $P \equiv_{\langle \mathcal{H},\mathcal{B}\rangle} Q$ does not hold. □

5 Special Cases

In this section, we analyze how $\langle \mathcal{H}, B\rangle$-models behave for special instantiations of $\mathcal{H}$ and $\mathcal{B}$. We first consider the case where either $\mathcal{H} = \mathcal{U}$ or $\mathcal{B} = \mathcal{U}$. We call the former scenario *body-relativized* and the latter *head-relativized*. Then, we sketch more general settings where the only restriction is that either $\mathcal{H} \subseteq \mathcal{B}$ or $\mathcal{B} \subseteq \mathcal{H}$ holds.

5.1 Body-Relativized and Head-Relativized Equivalence

First, we consider $\langle \mathcal{U}, \mathcal{B}\rangle$-equivalence problems, where $\mathcal{U}$ is fixed by the universe, but $\mathcal{B}$ can be arbitrarily chosen. Note that $\langle \mathcal{U}, \mathcal{B}\rangle$-equivalence ranges from strong (setting $\mathcal{B} = \mathcal{U}$) to uniform equivalence (setting $\mathcal{B} = \emptyset$ and cf. Corollary 1) and thus provides a common view on these two important problems, as well as on problems "inbetween" them. Second, head-relativized equivalence problems, $P \equiv_{\langle \mathcal{H},\mathcal{U}\rangle} Q$, have as special cases once more strong equivalence (now by setting $\mathcal{H} = \mathcal{U}$) but also the case where $\mathcal{H} = \emptyset$ is of interest, since it amounts to check whether P and Q possess the same answer sets under any addition of constraints. It is quite obvious that this holds iff P and Q are ordinarily equivalent, since constraints can only "rule out" answer sets. That observation is also reflected in Corollary 1, since the only unary program in $\mathcal{C}_{\langle \emptyset,\mathcal{U}\rangle}$ is the empty program.

The following result simplifies the definition of $\preceq^{\mathcal{B}}_{\mathcal{H}}$ within these settings.

Proposition 3. *For interpretations $V, Z \subseteq \mathcal{U}$ and an alphabet $\mathcal{A} \subseteq \mathcal{U}$, it holds that (i) $V \preceq^{\mathcal{A}}_{\mathcal{U}} Z$ iff $V \subseteq Z$ and $V|_{\mathcal{A}} = Z|_{\mathcal{A}}$; and (ii) $V \preceq^{\mathcal{U}}_{\mathcal{A}} Z$ iff $Z \subseteq V$ and $V|_{\mathcal{A}} = Z|_{\mathcal{A}}$.*

Thus, maximizing wrt $\preceq^{\mathcal{B}}_{\mathcal{H}}$ becomes in case of $\mathcal{H} = \mathcal{U}$ a form of $\subseteq$-maximization; and in case of $\mathcal{B} = \mathcal{U}$ a form of $\subseteq$-minimization. Obviously, both neutralize themselves for $\mathcal{B} = \mathcal{H} = \mathcal{U}$, i.e., in the strong equivalence setting, where $V \preceq^{\mathcal{U}}_{\mathcal{U}} Z$ iff $V = Z$.

For body-relativized equivalence, our characterization now simplifies as follows.

Corollary 2. *A pair (X, Y) of interpretations is an $\langle\mathcal{U}, \mathcal{B}\rangle$-model of $P \in \mathcal{C}_{\mathcal{U}}$ iff $X \subseteq Y$, $Y \models P$, $X \models P^Y$, and for all X' with $X \subset X' \subset Y$ and $X'|_{\mathcal{B}} = X|_{\mathcal{B}}$, $X' \not\models P^Y$.*

Observe that for the notions inbetween strong and uniform equivalence the maximality test, which tests if each X' with $X \subset X' \subset Y$ and $X'|_{\mathcal{B}} = X|_{\mathcal{B}}$ yields $X' \not\models P^Y$, gets more localized the more atoms are contained in $\mathcal{B}$. In particular, for $\mathcal{B} = \mathcal{U}$ it disappears and we end up with a very simple condition for $\langle\mathcal{U},\mathcal{U}\rangle$-models which exactly matches the definition of SE-models by Turner [11]: a pair (X, Y) of interpretations is an SE-model of a program P iff $X \subseteq Y$, $Y \models P$, and $X \models P^Y$.

For $\mathcal{B} = \emptyset$, on the other hand, we observe that $X'|_{\mathcal{B}} = X|_{\mathcal{B}}$ always holds for $\mathcal{B} = \emptyset$. Thus, a pair (X, Y) is a $\langle\mathcal{U}, \emptyset\rangle$-model of a program P, if $X \subseteq Y$, $Y \models P$, $X \models P^Y$, and for all X' with $X \subset X' \subset Y$, $X' \not\models P^Y$. These conditions are now exactly the ones given for UE-models following [2]. Hence, Corollary 2 provides a common view on the characterizations of uniform and strong equivalence.

For head-relativized equivalence notions, simplifications are as follows.

Corollary 3. *A pair (X, Y) of interpretations is an $\langle\mathcal{H},\mathcal{U}\rangle$-model of $P \in \mathcal{C}_{\mathcal{U}}$ iff $X \subseteq Y$, Y is an $\mathcal{H}$-total model for P, $X \models P^Y$, and for each $X' \subset X$ with $X'|_{\mathcal{H}} = X|_H$, $X' \not\models P^Y$.*

In the case of $\mathcal{H} = \mathcal{U}$, $\langle\mathcal{H},\mathcal{U}\rangle$-models again reduce to SE-models. The other special case is $\mathcal{H} = \emptyset$. Recall that $\langle\emptyset,\mathcal{U}\rangle$-equivalence amounts to ordinary equivalence. $\langle\emptyset,\mathcal{U}\rangle$-models thus characterize answer sets as follows: First, Y is an $\emptyset$-total model for P, iff no $X \subset Y$ satisfies $X \models P^Y$. Moreover, this requires that all $\langle\emptyset,\mathcal{U}\rangle$-models are total. So, the condition in Corollary 3 for $X \subset Y$ is immaterial and we have a one-to-one correspondence between $\langle\emptyset,\mathcal{U}\rangle$-models and answer sets of a program.

5.2 $\mathcal{B} \subseteq \mathcal{H}$ - and $\mathcal{H} \subseteq \mathcal{B}$ - Equivalence

Due to lack of space, we just highlight a few results here, in order to establish a connection between $\langle\mathcal{H}, \mathcal{B}\rangle$-models and relativized SE- and UE-models, as defined in [12].

Proposition 4. *For interpretations $V, Z \subseteq \mathcal{U}$ and alphabets $\mathcal{H}, \mathcal{B} \subseteq \mathcal{U}$ with $\mathcal{B} \subseteq \mathcal{H}$ (resp., $\mathcal{H} \subseteq \mathcal{B}$), $V \preceq^{\mathcal{B}}_{\mathcal{H}} Z$ iff $V|_{\mathcal{H}} \subseteq Z|_{\mathcal{H}}$ and $V|_{\mathcal{B}} = Z|_{\mathcal{B}}$ (resp., iff $Z|_{\mathcal{B}} \subseteq V|_{\mathcal{B}}$ and $V|_{\mathcal{H}} = Z|_{\mathcal{H}}$). Moreover, if $\mathcal{A} = \mathcal{H} = \mathcal{B}$, $V \preceq^{\mathcal{B}}_{\mathcal{H}} Z$ iff $V|_{\mathcal{A}} = Z|_{\mathcal{A}}$.*

Observe that $\preceq^{\mathcal{A}}_{\mathcal{A}}$-maximality (in the sense of Definition 8) of a pair (X, Y) for P reduces to test $X \models P^Y$. Thus, to make $(X|_{\mathcal{A}}, Y)$ an $\langle\mathcal{A}, \mathcal{A}\rangle$-model of P, we just additionally need $\mathcal{A}$-totality of Y. In other words, we obtain the following criteria.

Corollary 4. *Given $\mathcal{A} \subseteq \mathcal{U}$, a pair (X, Y) of interpretations is an $\langle\mathcal{A}, \mathcal{A}\rangle$-model of a program $P \in \mathcal{C}_{\mathcal{U}}$, iff (1) $X = Y$ or $X \subset Y|_{\mathcal{A}}$, (2) $Y \models P$ and for each $Y' \subset Y$, $Y' \models P^Y$ implies $Y'|_{\mathcal{A}} \subset Y|_{\mathcal{A}}$; and (3) if $X \subset Y$ then there exists an $X' \subseteq Y$ with $X'|_{\mathcal{A}} = X$, such that $X' \models P^Y$.*

This exactly matches the definition of $\mathcal{A}$-SE-models according to [12]. Finally, if we switch from $\langle\mathcal{A},\mathcal{A}\rangle$-equivalence to $\langle\mathcal{A},\emptyset\rangle$-equivalence (i.e., from relativized strong to relativized uniform equivalence) we obtain the following result for $\langle\mathcal{A},\emptyset\rangle$-models which coincides with an explicit definition of $\mathcal{A}$-UE-models according to [12].

Corollary 5. *Given $\mathcal{A} \subseteq \mathcal{U}$, a pair (X, Y) of interpretations is an $\langle\mathcal{A},\emptyset\rangle$-model of $P \in \mathcal{C}_{\mathcal{U}}$, iff (1) and (2) from Corollary 4 hold, and if $X \subset Y$ then there exists $X' \subseteq Y$ such that $X'|_{\mathcal{A}} = X$, $X' \models P^Y$, and for each $X'' \subset Y$ with $X'|_{\mathcal{A}} \subset X''|_{\mathcal{A}}$, $X'' \not\models P^Y$.*

6 Computational Issues

Former results on uniform [2] or relativized [12] equivalence show that these problems are, in general, Π_2^P-hard for disjunctive logic programs. Hence, $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence is Π_2^P-hard as well. However, Π_2^P-membership still holds in the view of Corollary 1. In particular, it is sufficient to guess an interpretation Y and a unary program $R \in \mathcal{C}_{\langle\mathcal{H},\mathcal{B}\rangle}$, and then to check whether Y is contained in either $\mathcal{AS}(P \cup R)$ or $\mathcal{AS}(Q \cup R)$, but not in both. Answer-set checking is in coNP, and since one can safely restrict Y and R to contain only atoms which also occur in P or Q, this algorithm for disproving $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence runs in nondeterministic polynomial time with access to an NP-oracle. Thus, that problem is in Σ_2^P, and consequently $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence is in Π_2^P.

Concerning implementation, we briefly discuss an approach which makes use of Corollary 1 in a similar manner and compiles $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence into ordinary equivalence for which a dedicated system exists [9]; a similar method was also discussed in [12, 10]. The idea hereby is to incorporate the guess of the unary context programs over the specified alphabets in both programs accordingly. To this end, let, for an $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence problem between programs P and Q, f as well as $c_{a,b}$ and $\bar{c}_{a,b}$ for each $a \in \mathcal{H}$, $b \in \mathcal{B} \cup \{f\}$, be new distinct atoms, not occurring in $P \cup Q$. Then, $P \equiv_{\langle\mathcal{H},B\rangle} Q$ holds iff $P^+_{\langle\mathcal{H},B\rangle}$ and $Q^+_{\langle\mathcal{H},B\rangle}$ are ordinarily equivalent, where, for $R \in \{P, Q\}$,

$$R^+_{\langle\mathcal{H},B\rangle} = R \cup \Big\{ c_{a,b} \vee \bar{c}_{a,b} \leftarrow;\ a \leftarrow b, c_{a,b} \mid a \in \mathcal{H}, b \in \mathcal{B} \cup \{f\} \Big\} \cup \{f \leftarrow\}.$$

In fact, the role of atoms $c_{a,f}$ is to guess a set of facts $F \subseteq \mathcal{H}$, while atoms $c_{a,b}$ with $b \neq f$ guess a subset of unary rules $a \leftarrow b$ with $a \in \mathcal{H}$ and $b \in \mathcal{B}$.

7 Conclusion

The aim of this work is to provide a general and uniform characterization for different equivalence problems, which have been handled by inherently different concepts, so far. We have introduced an equivalence notion parameterized by two alphabets to restrict the atoms allowed to occur in the heads, and respectively, bodies of the context programs. We showed that our approach captures the most important equivalence notions studied, including strong and uniform equivalence as well as relativized notions thereof.

Figure 1 gives an overview of $\langle\mathcal{H},\mathcal{B}\rangle$-equivalence and its special cases, i.e., relativized uniform equivalence (RUE), relativized strong equivalence (RSE), body-relativized equivalence (BRE), and head-relativized equivalence (HRE). On the bottom line

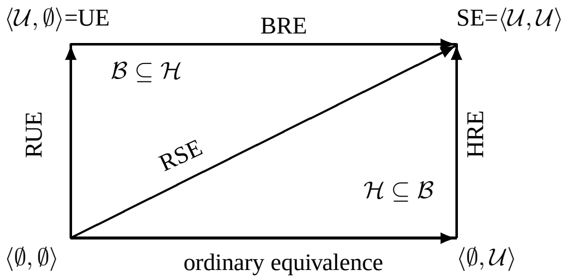

Fig. 1. The landscape of $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence with either $\mathcal{H} \subseteq \mathcal{B}$ or $\mathcal{B} \subseteq \mathcal{H}$.

we have ordinary equivalence, while the top-left corner amounts to uniform equivalence (UE) and the top-right corner to strong equivalence (SE).

Future work includes the study of further properties of $\langle \mathcal{H}, \mathcal{B} \rangle$-equivalence, as well as potential applications, which include relations to open logic programs [1] and new concepts for program simplification [3]. Also an extension in the sense of [5], where a further alphabet is used to specify the atoms which have to coincide in comparing the answer sets is considered. While [5] provides a characterization for relativized *strong* equivalence with projection, recent work [8] addresses the problem of relativized *uniform* equivalence with projection. Our results may be a basis to provide a common view on these two recent characterizations, as well.

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