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Bidual Horn Functions and Extensions*

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Abstract

Partially defined Boolean functions (pdBf) (T, F), where $T, F \subseteq \{0,1\}^n$ are disjoint sets of true and false vectors, generalize total Boolean functions by allowing that the function values on some input vectors are unknown. The main issue with pdBfs is the extension problem, which is deciding, given a pdBf, whether it is interpolated by a function f from a given class of total Boolean functions, and computing a formula for f. In this paper, we consider extensions of bidual Horn functions, which are the Boolean functions f such that both f and its dual function f^d are Horn. They are intuitively appealing for considering extensions because they give a symmetric role to positive and negative information (i.e., true and false vectors) of a pdBf, which is not possible with arbitrary Horn functions. Bidual Horn functions turn out to constitute an intermediate class between positive and Horn functions which retains several benign properties of positive functions. Besides the extension problem, we study recognition of bidual Horn functions from Boolean formulas and properties of normal form expressions. We show that finding a bidual Horn extension and checking biduality of a Horn DNF is feasible in polynomial time, and that the latter is intractable from arbitrary formulas. We also give characterizations of shortest DNF expressions of a bidual Horn function f and show how to compute such an expression from a Horn DNF for f in polynomial time; for arbitrary Horn functions, this is NP-hard. Furthermore, we show that a polynomial total algorithm for dualizing a bidual Horn function exists if and only if there is such an algorithm for dualizing a positive function.

Keywords: Boolean functions, Horn formulas, satisfiability, partially defined Boolean functions, characteristic models, polynomial algorithms

1 Introduction

The concept of a partially defined Boolean function (pdBf) [5, 8] is a natural generalization of the familiar concept of a Boolean function, by allowing that the function values on some input vectors are unknown. Those pdBfs have many applications, in particular in computer science and knowledge engineering.

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For example, a classical application of pdBfs is in the design of switching circuits. A customary method in that field is to specify the inputs on which the circuit must output 1 and the inputs on which it must output 0; the output on the remaining inputs remains unspecified and is considered as "don't care". Another application of pdBfs is with the representation of incomplete information about cause-effect relationships [8]. E.g., the effect of a number of facts (e.g., a patient is male, is a smoker etc.) on a specific disease (e.g., cancer) can be modeled as a Boolean function $f(x_1, x_2, \ldots, x_n)$, where the arguments x_i represent presence of the facts, and the value of f tells whether the disease is present or not. Since in general the results of all combinations of the facts on the disease will hardly be known, the relationship can be properly modeled by a pdBf. Furthermore, pdBfs have applications in machine learning. E.g., consider concept learning [1, 2, 32] in the following setting: Given a language of n Boolean valued attributes, find a hypothesis for their correlation, i.e., a function g from a fixed class of Boolean functions $\mathcal{C}$, that accurately approximates the actual correlation, which is an f in $\mathcal{C}$, after seeing a reasonably small number of examples, i.e., values of f on particular vectors selected by the learning algorithm. In our terms, the algorithm gradually refines a pdBf until finally a total Boolean function is output. In this context, it is interesting to know whether the pdBf given by the considered examples implicitly defines a function from $\mathcal{C}$; if this is recognized, the algorithm can stop and output g which describes the exact correlation.

More formally, a pdBf is a pair (T, F) of sets T and F of true and false vectors in $\{0,1\}^n$, respectively, where $T \cap F = \emptyset$. Clearly, each pdBf can be completed to some total Boolean function f. In general, however, one is interested to know whether this is possible for some f from a particular class $\mathcal{C}$ of Boolean functions, i.e., whether an *extension* f in $\mathcal{C}$ exists such that $T \subseteq T(f)$ and $F \subseteq F(f)$, where $T(f)$ (resp., $F(f)$) denotes the set of true (resp., false) vectors of f. This is known as the *extension problem*, and corresponds in a sense to the satisfiability problem of Boolean formulas.

The extension problem and variants thereof have been investigated for a number of classes of Boolean functions [8, 6, 5, 28]. Among these classes are Horn functions, which are of central interest in many domains. A function is *Horn* if it can be represented by a DNF (disjunctive normal form) in which each term contains at most one negative literal. It is well-known that the Horn functions f are those whose set $F(f)$ of false vectors is closed under intersection (see Section 2); they play an important role in artificial intelligence, logical databases, and logic in computer science, cf. [16, 7, 21]. As shown in [28, 6], a Horn extension of a pdBf can be found in polynomial time. In fact, a Horn extension for (T, F) exists precisely if the true vectors T are disjoint from the closure of the false vectors F under intersection. However, this characterization shows that the Horn extension problem is, in a sense, asymmetric in the input T and F. From a conceptual point, we could ask for a more balanced role of T and F in the condition for a Horn extension. Thus, we might search for suitable additional constraints to reach this goal.

A natural and suggestive possibility at hand is to require a dual behavior between T and F, since 0 and 1 are dual values. This leads to the concept of bidual Horn functions: A function f is *bidual Horn*, if $F(f)$ is closed under intersection and, dually, $T(f)$ is closed under union (i.e., under disjunction of vectors); that is, both f and its dual f^d are Horn.

Observe that besides bidual Horn functions, other possibilities for balancing the role of T and F exist. E.g., in [14] the class of submodular functions has been investigated, where a function f is submodular if f and its contra-dual are Horn, and in [13] the class of double Horn functions, where a function f is double Horn if f and is complement are Horn.

It turns out that Bidual Horn functions have interesting properties. Firstly, from the logical perspective, the bidual Horn functions are those functions f such that $F(f)$ is described by logical implications

$$x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \rightarrow x_{i_0}, \tag{1.1}$$

and, dually, $T(f)$ by

$$\overline{x}_{i_1} \wedge \overline{x}_{i_2} \wedge \cdots \wedge \overline{x}_{i_k} \rightarrow \overline{x}_{i_0} \tag{1.2}$$

where both the antecedent and the consequent may be empty. Thus, if the true vectors are seen as legal state descriptions, then they are fully characterized by dependencies of literals from false facts, and the illegal states are characterized by similar dependencies of literals from true facts. This property is preserved if truth and falsity are interchanged.

Secondly, the bidual Horn functions constitute an intermediate class between the classes of positive functions and Horn functions, which retains many of the benign properties of positive functions. In particular, apart from syntactical and semantical properties, certain important computational problems which are intractable for Horn functions are for bidual Horn functions, like for positive functions, polynomial (see below). Most importantly, any irredundant prime DNF of a bidual Horn function contains the same number of term, and the computation of a shortest DNF expression from an arbitrary Horn DNF, which is NP-hard for arbitrary Horn functions, is proved to be polynomial. Observe that few similar subclasses of Horn functions are known; e.g., the class of quasi-acyclic Horn functions [20], which is incomparable to the class of bidual Horn functions. Since bidual Horn functions are polynomial-time recognizable, this means that our algorithms can be added to a tool-box for tractable recognition and handling of important problems on Horn functions.

The main contributions of this paper can be shortly summarized as follows.

- We introduce the class of bidual Horn functions, $\mathcal{C}_{BH}$, and investigate their properties. In particular, we present characterizations of bidual Horn functions in terms of their prime implicants, and we characterize the shortest DNF expressions of a bidual Horn function f, measured on the one hand by the smallest number of terms (term-shortest DNF) in a DNF for f, or, on the other hand, by the smallest number of literals (literal-shortest DNF), which corresponds to the length of the DNF. Based on this, we develop polynomial time algorithms for recognizing a bidual Horn function from a given Horn DNF, as well as for computing a term-shortest or literal-shortest DNF. These are positive results, since computing a term-shortest or a literal-shortest DNF of an arbitrary Horn DNF are well-known NP-hard problems [3, 27, 18].

- We present an algorithm that decides the existence of a bidual Horn extension f for a given partially defined Boolean function (T, F) in $O(n|T||F|)$ time, where n is the dimension of the Boolean vectors. and describe how a Horn DNF for such an f can be output in $O(n|T|(|T| + |F|))$ time. Moreover, we show that finding term-shortest or literal-shortest bidual Horn extensions in DNFs are NP-hard problems.

- We address the problem of computing all bidual Horn extensions of (T, F), and give evidence that a procedure for enumerating the bidual Horn extensions $\varphi_1, \varphi_2, \ldots$ of (T, F) with polynomial delay between subsequent outputs is hard to find. In fact, we show that given a bidual Horn extension φ of (T, F), deciding whether another bidual Horn extension $\psi \not\equiv \varphi$ exists is at least as hard as the positive duality problem [4], i.e., given two positive DNFs φ, ψ, decide whether ψ represents the dual of the function represented by φ. The positive duality problem and equivalent problems have been tackled by many researchers, but no polynomial algorithm is known [23, 4, 15, 11, 24]. This strongly supports that a polynomial time algorithm for the unique bidual Horn extension problem, i.e., deciding whether a pdBf (T, F) implicitly defines a total bidual Horn function is difficult to find.

• We study transformation problems between different representations for bidual Horn functions, in particular (Horn) DNF formulas and characteristic set [25, 24, 26] (or bases [10]), which are a vector-based representation of arbitrary Horn functions that has received much interest in the context of knowledge representation and reasoning (see Section 6 for details). We show that the transformation between a Horn DNF of f and its (unique) characteristic set can be done in polynomial time, i.e., given a Horn DNF of f, the characteristic set of f is constructible in polynomial time and vice versa. Furthermore, we show that several transformations between representations of f and its dual f^d are polynomial-time equivalent to the well-known problem of dualizing a positive function [4, 11, 15]. Namely, the transformation between (i) the characteristic set of f and a Horn DNF of f^d; $(ii)$the characteristic set of f and the characteristic set of f^d, and (iii) a Horn DNF of f and a Horn DNF of f^d, i.e., dualization of a bidual Horn function. This can be seen as a positive result, because it is believed that for an arbitrary Horn function f, the transformation between the characteristic set of f^d and an Horn DNF of f is strictly harder than the problem of dualizing a positive function [24].

The remainder of this paper is structured as follows. In the next section, we recall some basic concepts, fix notations, and formulate major computational problems on Boolean functions. In Section 3, we introduce bidual Horn functions and study recognition from a formula. Issues on bidual Horn extensions are considered in Section 4. In Section 5, we turn our attention to shortest DNF expressions of bidual Horn functions and shortest bidual Horn extensions. In Section 6, we consider the transformation problems for bidual Horn functions. In the final Section 7, we address further issues and conclude the paper. Some proofs are omitted; they can be found in [12].

2 Preliminaries

We use letters a, b, c and u, v, w to denote vectors in $\{0, 1\}^n$, and use $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$. In general, we also allow $n = 0$; the set $\{0, 1\}^0$ contains a single vector, which is the empty vector $()$. As usual, $v \wedge w$ (resp., $v \vee w$) denotes the *intersection* (resp., *union*) (i.e., the componentwise conjunction (resp., disjunction)) of vectors v and w; e.g., if $v = (1100)$ and $w = (1010)$, then $v \wedge w = (1000)$ and $v \vee w = (1110)$.

For each $v = (v_1, v_2, \ldots, v_n)$ we define $ON(v) = \{i \mid v_i = 1\}$ and $OFF(v) = \{i \mid v_i = 0\}$, and denote $\overline{v} = (\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n)$, where $\overline{v}_i = 1 - v_i$, $i = 1, 2, \ldots, n$. Moreover, for every $I \subseteq \{1, \ldots, n\}$, we denote by x^I its characteristic vector, implicitly defined by $ON(x^I) = I$; e.g., if $n = 5$ and $I = \{1, 3\}$, then $x^I = (10100)$.

Let $S \subseteq \{0,1\}^n$ be a set of vectors. Then, $\bigwedge S = \bigwedge_{v\in S} v$ and $\bigvee S = \bigvee_{v\in S} v$ denote the simultaneous intersection and union, respectively of all vectors in S; in particular, $\bigwedge \emptyset = \mathbf{1}$ and $\bigvee \emptyset = \mathbf{0}$. Moreover, $Cl_\wedge(S)$ (resp. $Cl_\vee(S)$) denotes the closure of S under intersection $v \wedge w$ (resp., union $v \cup w$) of vectors v and w, called the *intersection* (*resp., union*) *closure of* S. For a subset $I \subseteq \{1, 2, \dots, n\}$, $S[I]$ denotes the *projection* of S to I.

Example 2.1 *Let* $S = \{(0101), (1001), (1000)\}$. *Then* $\bigwedge S = \{(0000)\}$, $\bigvee S = \{(1101)\}$, $Cl_\wedge(S) = \{(0101), (1001), (1000), (0001), (0000)\}$, *and* $Cl_\vee(S) = \{(0101), (1001), (1000), (1101)\}$. *For* $I = \{1, 3\}$, *we have* $S[I] = \{(00), (10)\}$. □

Recall that a *Boolean function*, or a *function* in short, is a mapping $f : \{0,1\}^n \to \{0,1\}$. The sets $T(f) = \{v \mid f(v) = 1\}$ and $F(f) = \{v \mid f(v) = 0\}$ are the true vectors and false vectors of f, respectively. Notice that for $n = 0$, there are precisely two Boolean functions, $f = \bot$ and $f = \top$, which correspond to truth and falsity, respectively. For any function f, we denote by $\overline{f}$ and f^d its *negation* (or *complement*) and *dual* respectively, which are defined by $T(\overline{f}) = F(f)$ and $T(f^d) = \{a \mid \overline{a} \in F(f)\}$. Note that $f^d = \overline{g}$, where $T(g) = \{a \mid \overline{a} \in T(f)\}$.

A *partially defined Boolean function* (pdBf) is a mapping $p : T \cup F \mapsto \{0,1\}$ defined by $p(v) = 1$ if $v \in T$; 0 if $v \in F$, where $T \subseteq \{0,1\}^n$ denotes a set of true vectors (or positive examples) and $F \subseteq \{0,1\}^n$ denotes a set of false vectors (or negative examples) such that $T \cap F = \emptyset$. For simplicity, a pdBf is denoted by a pair of sets (T, F). A pdBf is called *total* if $T \cup F = \{0,1\}^n$.

Notice that (T, F) can be seen as a representation for all Boolean functions f such that $T(f) \supseteq T$ and $F(f) \supseteq F$; any such f is called an *extension* of (T, F). The main issue in the context of pdBf concerns existence and properties of extensions, subject to the condition that they are from a certain class of Boolean functions [6, 28].

There is a vast literature on classes of Boolean functions and their properties, in particular on computational aspects [33], among which the classes $\mathcal{C}_\leq$ of positive functions and $\mathcal{C}_{Horn}$ of Horn functions are most well-known. A function f is *positive* (also called *monotone*) if $v \leq w$ implies $f(v) \leq f(w)$, where $\leq$ is componentwise and $0 \leq 1$. A *Horn* function f has the well-known algebraic characterization

$$f(v \wedge w) \leq f(v) \vee f(w),$$

which is equivalent to $F(f) = Cl_\wedge(F(f))$.

Equivalent definitions of positive and Horn functions can be given in terms of *disjunctive normal form* (DNF). We assume that Boolean *variables* are from $x_1, x_2, \dots, x_n$. A *literal* L is either a variable x_i or its complement $\overline{x}_i$, which are respectively referred to as *positive* and *negative* literals. A *term* t is a conjunction $\bigwedge_{i\in P(t)} x_i \wedge \bigwedge_{j\in N(t)} \overline{x}_j$ of literals such that $P(t) \cap N(t) = \emptyset$. We often omit conjunction symbols if no confusion arises. The empty term (representing truth) with $P(t) = N(t) = \emptyset$ is denoted by $\top$. Let $V(t) = P(t) \cup N(t)$ denote the variable indices in t. A DNF φ is a *disjunction* $\bigvee_{i=1}^k t_i$ of terms; the empty DNF (representing falsity) is denoted by $\bot$. The length of a DNF (or arbitrary formula) φ, denoted by $|\varphi|$, is the number of symbols in φ. A term t is *positive* if $N(t) = \emptyset$, *Horn* if $|N(t)| \leq 1$, and *pure Horn* if $|N(t)| = 1$. A DNF $\varphi = \bigvee_i t_i$ is called *positive* if all t_i are positive, *Horn* if all t_i are Horn, and *pure Horn* if all t_i are pure Horn.

Example 2.2 *For example, $t_1 = x_1x_2x_4$, $t_2 = x_1x_4\overline{x}_5x_6$ and $t_3 = x_2\overline{x}_3\overline{x}_5$ are terms, while $t_4 = x_2x_4\overline{x}_2$ is not; t_1 is positive (and hence Horn) and has $P(t_1) = \{1, 2, 4\}$ and $N(t_1) = \emptyset$, t_2 is (pure) Horn and has $P(t_2) = \{1, 4, 6\}$ and $N(t_2) = \{5\}$, and t_3 is neither positive and Horn. The DNFs $\varphi^{(1)} = x_2 \vee x_1x_3 \vee x_1x_4$, $\varphi^{(2)} = \overline{x}_2 \vee \overline{x}_1x_3 \vee x_3\overline{x}_4$ and $\varphi^{(3)} = x_2\overline{x}_3 \vee x_1x_3 \vee \overline{x}_2x_3$, respectively, are positive, pure Horn and Horn.* □

We call a function *positive* (resp., (*pure*) *Horn*) if and only if it can be represented by some positive (resp., (pure) Horn) DNF. It is known that these definitions using DNFs coincide with the above semantic definitions of positive and Horn functions.

A term t is an *implicant* of a formula φ (resp., function f) if $t \leq \varphi$ (resp., $t \leq f$) holds; here t and φ are regarded as functions they represent, and $f_1 \leq f_2$ denotes $T(f_1) \subseteq T(f_2)$. An implicant t is *prime* if no proper subterm of t is an implicant. A DNF $\varphi = \bigvee_i t_i$ is called *prime* if all terms t_i in φ are prime implicants, and *irredundant* if no DNF, which is obtained by dropping some terms t_i in φ, represents the same function. A prime implicant t of a function f is called *essential* if all prime DNFs representing f contain t. For example, a DNF $\varphi = x_1\overline{x}_2 \vee x_1\overline{x}_3 \vee x_2\overline{x}_3 \vee x_4$ is prime because all terms $x_1\overline{x}_2$, $x_1\overline{x}_3$, $x_2\overline{x}_3$ and x_4 are prime implicants, but it is not irredundant because $\varphi' = x_1\overline{x}_2 \vee x_2\overline{x}_3 \vee x_4$ represents the same function as φ. In this case, it can be shown that x_4 is essential.

Let t_1 and t_2 be terms such that $P(t_1) \cap N(t_2) = \{l\}$ and $N(t_1) \cap P(t_2) = \emptyset$. Then a term t_3 is called the *consensus* of the ordered pair t_1,t_2 if

$$P(t_3) = (P(t_1) \setminus \{l\}) \cup P(t_2) \text{ and } N(t_3) = N(t_1) \cup (N(t_2) \setminus \{l\}). \tag{2.1}$$

E.g., $x_2\overline{x}_3x_4\overline{x}_5x_6$ is the consensus of $x_1\overline{x}_3x_4\overline{x}_5$ and $\overline{x}_1x_2\overline{x}_3x_4x_6$. Note that, in this definition, the roles of t_1 and t_2 are asymmetric; t_1 (resp., t_2) is called the *left-parent* (resp., *right-parent*) of t_3, and t_3 is the *child* of t_1 as well as being the child of t_2. Let $\varphi = \bigvee_i t_i$ be an arbitrary DNF expression of a function f. It is known [30] that every prime implicant t of f can be derived from the terms in φ by applying a consensus procedure. In other words, there is a sequence $t^{(1)}, t^{(2)}, \ldots, t^{(m)} = t$ of terms such that each $t^{(k)}$ is either in φ (i.e., $t^{(k)} = t_i$ for some i) or the consensus of two terms $t^{(k_1)}$ and $t^{(k_2)}$ such that $k_1, k_2 < k$. Since the consensus of (pure) Horn terms t_1 and t_2 is (pure) Horn, the above statement implies that all prime implicants of a (pure) Horn function are (pure) Horn [19].

On the computational side, the following problems have been extensively studied for many classes $\mathcal{C}$ of Boolean functions:

(Recognition) : Given a formula φ, does the function represented by φ belong to $\mathcal{C}$?

(Extension) : Given a pdBf (T, F), does there exist an extension f of (T, F) such that $f \in \mathcal{C}$?

In case of the extension problem, one is usually also interested in a representation of f, e.g., by a formula φ. Variants of this problem concern inquiring the uniqueness of an extension, and generation of all extensions (i.e., representations thereof). We shall study the above two problems for the class of bidual Horn functions, which is introduced in the next section.

3 Bidual Horn Functions

We start with a formal definition of bidual Horn functions.

Definition 3.1 *Let f be Boolean function. Then, f is bidual Horn, if and only if $F(f) = Cl_\wedge(F(f))$ and $T(f) = Cl_\vee(T(f))$. The class of all bidual Horn functions is denoted by $\mathcal{C}_{BH}$.*

As a consequence, biduality is algebraically characterized by the two conjoined inequalities

$$\begin{aligned} f(x \wedge y) &\leq f(x) \vee f(y) \\ f(x) \wedge f(y) &\leq f(x \vee y). \end{aligned}$$

Equivalently, a function is bidual Horn if and only if both f and f^d are Horn. For example,

$$f = \overline{x}_1 x_2 x_3 \vee x_1 \overline{x}_3 x_4 \vee x_2 x_3 x_4$$

is bidual Horn, because

$$\begin{aligned} f^d &= (\overline{x}_1 \vee x_2 \vee x_3)(x_1 \vee \overline{x}_3 \vee x_4)(x_2 \vee x_3 \vee x_4) \\ &= \overline{x}_1 x_4 \vee x_1 x_2 \vee x_2 \overline{x}_3 \vee x_2 x_4 \vee x_1 x_3 \vee x_3 x_4. \end{aligned}$$

It is well-known that the dual f^d of any positive function f is positive as well. As a consequence, if f is positive, then both f and f^d are Horn; thus,

Proposition 3.1 *$\mathcal{C}_\leq \subset \mathcal{C}_{BH}$, i.e., the bidual Horn functions properly generalize positive functions.*

The first problem we address is recognition of bidual Horn functions from a given formula φ representing a function f. The definition of biduality implies a naive exponential algorithm which checks the intersection and union condition on $F(f)$ and $T(f)$, respectively. This algorithm is not much satisfactory, however, as it uses exponential space in the worst case. Algorithms in polynomial space are feasible, but a polynomial time algorithm is unlikely to exist, which is a consequence of the following result.

Theorem 3.2 *Let φ be a formula. Then deciding whether φ represents a bidual function is* co-NP-*complete, even if φ is a DNF.*

Proof. The problem is in co-NP, since a guess for vectors $v^{(1)}$ and $v^{(2)}$ such that either $(\varphi(v^{(1)} \wedge v^{(2)}) = 1$ and $\varphi(v^{(1)}) = \varphi(v^{(2)}) = 0)$ or $(\varphi(v^{(1)} \vee v^{(2)}) = 0$, $\varphi(v^{(1)}) = \varphi(v^{(2)}) = 1)$ can be verified in polynomial time. To show the hardness, we use the reduction from the problem of deciding whether a DNF ψ on variables $x_1, x_2, \ldots, x_n$ is a tautology (i.e., $\psi = \top$), which is well-known co-NP-complete [17]. Define $\varphi = \psi \vee x_{n+1} x_{n+2} \vee \overline{x}_{n+1} \overline{x}_{n+2}$. Obviously, φ is Horn if and only if ψ is a tautology. This is also equivalent to the condition that φ is a tautology, that is $\varphi = \top \in \mathcal{C}_{BH}$. Thus, φ is bidual Horn if and only if ψ is a tautology. □

However, the recognition problem is polynomial if the input formulas φ are restricted to Horn DNFs. We obtain this result from a useful characterization of bidual Horn functions presented next.

We introduce some additional notations. For a pair of terms t_i and t_j, let us denote by $t^+_{i,j}$ the positive term such that $P(t^+_{i,j}) = P(t_i) \cup P(t_j)$, and by $t^\pm_{i,j}$ the positive term such that $P(t^\pm_{i,j}) = V(t_i) \cup V(t_j)$. For example, if $t_1 = x_1\overline{x}_2x_3$ and $t_2 = \overline{x}_1x_4$, then $t^+_{1,2} = x_1x_3x_4$ and $t^\pm_{1,2} = x_1x_2x_3x_4$. Note that a term t is a Horn implicant of f if $t \le f$ and $|N(t)| \le 1$.

Lemma 3.3 *Let f be a Horn function. Then the following statements are equivalent:*

(i) *f is bidual.*

(ii) *For every pair of Horn implicants t_i and t_j of f that have different negative literals, i.e., $|N(t_i) \cup N(t_j)| = 2$, it holds that*

$$t^+_{i,j} \le f. \tag{3.1}$$

(iii) *For every pair of Horn implicants t_i and t_j of f such that $|N(t_i) \cup N(t_j)| = 2$, it holds that*

$$t^\pm_{i,j} \le f. \tag{3.2}$$

Proof. (i) $\Longrightarrow$ (ii): Assume there is a vector b such that $t^+_{i,j}(b) = 1$ and $f(b) = 0$ for some implicants t_i, t_j of f such that $|N(t_i) \cup N(t_j)| = 2$. Clearly, $ON(b) \supseteq P(t_i) \cup P(t_j)$ holds. Furthermore, $N(t_i) \cup N(t_j) \subseteq ON(b)$ holds, because otherwise $N(t_k) \subseteq OFF(b)$ implies $t_k(b) = 1$, and hence $f(b) = 1$, which is a contradiction. Now let $N(t_i) = \{h\}$ and $N(t_j) = \{l\}$, and take three vectors b, $b^{(h)}$ and $b^{(l)}$, where $b^{(k)}$ denotes the vector such that $ON(b^{(k)}) = ON(b) \setminus \{k\}$. Then,

$$b = (b^{(h)}) \vee (b^{(l)}).$$

Since $t_i(b^{(h)}) = t_j(b^{(l)}) = 1$, $f(b^{(h)}) = f(b^{(l)}) = 1$ holds, and $f(b) = 0$ holds by assumption. Hence $T(f)$ is not closed under union; i.e., f is not bidual Horn.

(ii) $\Longrightarrow$ (iii): Immediate from $t^\pm_{i,j} \le t^+_{i,j}$.

(iii) $\Longrightarrow$ (i): Assume that f is not bidual. Since f is Horn, there are three vectors u, $v^{(i)}$ and $v^{(j)}$ such that $u = v^{(i)} \vee v^{(j)}$, $f(u) = 0$ and $f(v^{(i)}) = f(v^{(j)}) = 1$. For $k = i, j$, $f(v^{(k)}) = 1$ implies that there is a Horn implicant $t_k = \bigwedge_{q\in P(t_k)} x_q \bigwedge_{q\in N(t_k)} \overline{x}_q$ of f such that $t_k(v^{(k)}) = 1$. Then

$$ON(u) \;=\; ON(v^{(i)}) \cup ON(v^{(j)}) \;\supseteq\; P(t_i) \cup P(t_j). \tag{3.3}$$

Furthermore, $N(t_i)$ and $N(t_j)$ satisfy the following conditions.

(a) $N(t_i), N(t_j) \neq \emptyset$ holds. Assume that $N(t_i) = \emptyset$. Then (3.3) implies $t_i(u) = 1$, i.e., $f(u) = 1$, which is a contradiction. The case $N(t_j) = \emptyset$ is analogous. Thus we have $|N(t_i)| = |N(t_j)| = 1$.

(b) $N(t_i) \subseteq ON(v^{(j)})$ and $N(t_j) \subseteq ON(v^{(i)})$ hold. Assume that $N(t_i) \subseteq OFF(u)$ holds. Then (3.3) implies $t_i(u) = 1$, i.e., $f(u) = 1$, which is a contradiction. The case $N(t_j) \subseteq OFF(u)$ is analogous.

(c) $N(t_i) \neq N(t_j)$ holds. Otherwise, (b) implies $N(t_i) = N(t_j) \subseteq ON(v^{(i)})$, which is a contradiction to $t_i(v^{(i)}) = 1$.

By (a) and (c), t_i and t_j are Horn implicants of f such that $|N(t_i) \cup N(T_j)| = 2$. By (3.3) and (b),

$$ON(u) = ON(v^{(i)}) \cup ON(v^{(j)}) \supseteq V(t_i) \cup V(t_j).$$

Thus u satisfies $f(u) = 0$ and $t^{\pm}_{i,j}(u) = 1$, which implies that (3.2) does not hold. □

Exploiting the following lemma, we obtain a polynomial time algorithm for checking the biduality of a Horn DNF.

Lemma 3.4 *Let φ be a Horn DNF. Then φ represents a bidual Horn function if and only if*

$$t^{+}_{i,j} \leq \varphi \qquad (\textit{equivalently, } t^{\pm}_{i,j} \leq \varphi) \tag{3.4}$$

holds for all pairs of Horn terms t_i and t_j in φ such that $|N(t_i) \cup N(t_j)| = 2$. □

Theorem 3.5 *Given a Horn DNF φ, deciding whether it represents a bidual Horn function can be done in $O(m^2|\varphi|)$ time, where m denotes the number of terms in φ.*

Proof. A straightforward procedure tests whether condition (3.4) holds for every pair of terms t_i and t_j in φ with a distinct negative literal. Each test $t^{+}_{i,j} \leq \varphi$ can be done in $O(|\varphi|)$ time [9, 22]. Totally, it requires $O(\binom{m}{2}|\varphi|) = O(m^2|\varphi|)$ time. □

4 Bidual Horn Extensions

In this section, we address the problem of finding bidual Horn extensions for partially defined Boolean functions. Recall that a partially defined Boolean function is a pair (T, F), of true vectors T and false vectors F. The extension problem, deciding whether (T, F) has a bidual Horn extension, is a relaxation of the problem of actually finding a bidual extension. Since usually a constructive algorithm for the extension problem gives rise to an algorithm for the latter, we first consider the extension problem.

Let us look at Horn functions for a moment. The existence of Horn extensions of a pdBf (T, F) is characterized by the following simple criterion, which can be checked in polynomial time: (T, F) has a Horn extension if and only if $T \cap Cl_{\wedge}(F) = \emptyset$ [28, 6]. Thus, the obvious necessary condition $T \cap Cl_{\wedge}(F) = \emptyset$ is also sufficient.

For bidual Horn functions, we obtain an analogous necessary condition for the existence of an extension: (T, F) has a bidual Horn extension only if $Cl_{\vee}(T) \cap Cl_{\wedge}(F) = \emptyset$.

It appears that checking this condition is expensive; as shown in [12], the test is co-NP-complete. Fortunately, intractability of the extension problem is not a consequence thereof, as this necessary condition is not sufficient in general, as shown by the following example.

Example 4.1 Let (T, F) be a pdBf defined by $T = \{a^{(1)} = (10100), a^{(2)} = (01010)\}$ and $F = \{b^{(1)} = (11111), b^{(2)} = (11100)\}$. It is easily checked that $Cl_\vee(T) \cap Cl_\wedge(F) = \emptyset$. However, (T, F) has no bidual Horn extension. Indeed, assume that (T, F) has such an extension f. Let t_1 and t_2 be Horn implicants of f such that $t_1(a^{(1)}) = 1$ and $t_2(a^{(2)}) = 1$, respectively. Then t_1 and t_2 satisfy the following:

(i) $N(t_1), N(t_2) \neq \emptyset$ holds, because otherwise (i.e., $N(t_k) = \emptyset$), then $t_k(b^{(1)}) = 1$, and hence $f(b^{(1)}) = 1$ holds, a contradiction. This means $|N(t_1)| = |N(t_2)| = 1$.

(ii) $N(t_1) = N(t_2)$ holds, because otherwise, $t^+_{1,2} \leq f$ holds by Lemma 3.3 (ii), but $t^+_{1,2}$ satisfies $t^+_{1,2}(b^{(1)}) = 1$, a contradiction to our assumption.

By (i) and (ii), t_1 and t_2 must have a common negative literal, which must be $\overline{x}_5$ in this case. Thus it follows $t_1(b^{(2)}) = 1$; but $b^{(2)} \in F$. □

Thus, the attempt to obtain a polynomial time algorithm for the bidual Horn extension problem from simple characterizations as in the case of Horn functions fails. Nonetheless, we can fortunately show that the problem is polynomial, and that a bidual Horn extension can be output in polynomial time. We need some further concepts.

For a pdBf (T, F), let us define

$$F^+(v) = \{w \in F \mid w \geq v\}, \qquad I(v) = ON(\bigwedge F^+(v)) \setminus ON(v). \tag{4.5}$$

In particular, if no $w \in F^+(v)$ exists, we have $I(v) = OFF(v)$. It is not difficult to verify that for every $v \in T$ such that $v \neq \mathbf{1}$, every Horn extension f of (T, F) must have an Horn implicant t such that $P(t) = ON(v)$ and $N(t) \subseteq I(v)$.

We denote for every $v \in T$ by $R(v)$ the set of all Horn terms t_v such that (i) $P(t_v) = ON(v)$, and (ii) $\emptyset \neq N(t_v) \subseteq I(v)$ if $v \neq \mathbf{1}$, and $N(t_v) = \emptyset$ if $v = \mathbf{1}$ (i.e., $t_{\mathbf{1}} = x_1 \cdots x_n$); every term $t_v \in R(v)$ is called *canonical* for v (with respect to (T, F)). For a pdBf (T, F), $\varphi = \bigwedge_{v \in T} t_v$, where $t_v \in R(v)$ is called *canonical* Horn DNF of (T, F). Since any canonical Horn DNF of (T, F) represents an extension of (T, F), a Boolean function represented by a canonical Horn DNF is called *canonical* Horn extension of (T, F).

Example 4.2 *Consider the pdBf* (T, F)*, where* $T = \{(0010), (0110)\}$ *and* $F = \{(1001), (1010), (1100), (1011), (1101)\}$*. Let* $v^{(1)} = (0010) \in T$ *and* $v^{(2)} = (0110) \in T$*. Then,*

$$\begin{aligned} F^+(v^{(1)}) &= \{(1010), (1011)\} & F^+(v^{(2)}) &= \emptyset \\ I(v^{(1)}) &= ON(1010) \setminus ON(0010) = \{1\} & I(v^{(2)}) &= ON(1111) \setminus ON(0110) = \{1, 4\} \\ R(v^{(1)}) &= \{x_3\overline{x}_1\} & R(v^{(2)}) &= \{x_2x_3\overline{x}_1, x_2x_3\overline{x}_4\} \end{aligned}$$

The DNFs $\varphi_1 = x_3\overline{x}_1 \vee x_2x_3\overline{x}_1$ and $\varphi_2 = x_3\overline{x}_1 \vee x_2x_3\overline{x}_4$ are the canonical Horn DNFs of (T, F); they are indeed Horn extensions of (T, F). □

The following lemma is the key to our algorithm for finding a bidual Horn extension.

Lemma 4.1 *A pdBf (T, F) has a bidual Horn extension if and only if there exists a canonical term $t_v \in R(v)$ for every $v \in T$, such that any term $t^+_{v,w}$ defined from a pair of terms t_v and t_w with $|N(t_v) \cup N(t_w)| = 2$ satisfies $T(t^+_{v,w}) \cap F = \emptyset$. Furthermore, given such a choice t_v for $v \in T$, the DNF*

$$\varphi = \bigvee_{v \in T} t_v \vee \bigvee_{\substack{v,w \in T: \\ |N(t_v) \cup N(t_w)|=2}} t^+_{v,w} \tag{4.6}$$

represents a bidual Horn extension of (T, F).

Proof. For the only-if-part, let f be a bidual Horn extension of (T, F). Since f is Horn, for every $v \in T$, there is some canonical term $t_v \in R(v)$ such that $t_v \leq f$. Fix a choice of such terms t_v. For each pair of terms t_v and t_w with $|N(t_1) \cup N(t_2)| = 2$, $t^+_{v,w} \leq f$ holds by Lemma 3.3 (ii). Now consider the DNF φ of (4.6); it represents a Horn function f_φ such that $f_\varphi \leq f$. Clearly, $T(f_\varphi) \supseteq T$ and $F(f_\varphi) \supseteq F(f) \supseteq F$. Therefore, f_φ is an extension of (T, F). Moreover, Lemma 3.4 implies that f_φ is bidual.

Let us then show the if-part. By Lemma 3.4, the DNF φ of (4.6) represents a bidual Horn function f_φ. From the definitions, we obtain that $T(f_\varphi) \cap F = \emptyset$ and $T(f_\varphi) \supseteq T$. Thus, φ represents a bidual Horn extension of (T, F). □

From this lemma, a straightforward algorithm for finding a bidual Horn function which has $O(n|T|^2|F|)$ time complexity algorithm can be derived. Exploiting duality, however, we can find a faster algorithm. First, we introduce some additional concepts. A term t is called *co-Horn* if $|P(t)| \leq 1$. A DNF $\varphi = \bigvee_i t_i$ is called *co-Horn* if every term t_i is co-Horn; a Boolean function is co-Horn if it can be represented by some co-Horn DNF. Notice that f is co-Horn if and only if $f^* = f(\bar{x})$ is Horn, and that co-Horn functions have properties that are dual to the properties of Horn functions. For example, the set $F(f)$ of a co-Horn function f is closed under union, opposed to closedness of under intersection. Therefore, for each $v \in T(f)$, a unique maximal $w \in F(f)$ exists such that $w \leq v$.

Note that a pdBf (T, F) has a bidual Horn extension if and only if a pdBf (T, F) has a Horn extension f and a reverse pdBf (F, T) has a co-Horn extension g such that $\overline{f} = g$. Thus in the context of pdBfs (F, T) instead of (T, F), we define the canonical concept for co-Horn functions in a dual way. For $w \in F$, let us define

$$F^-(w) = \{v \in T \mid v \leq w\}, \qquad \text{co-}I(w) = OFF(\bigvee F^-(w)) \setminus OFF(w), \tag{4.7}$$

and let co-$R(w)$ be the set of canonical co-Horn terms of w, which are all terms t such that $N(t) = OFF(w)$ and $P(t) = \{\ell\}$ with $\ell \in$co-$I(w)$ for $v \neq \mathbf{0}$ and the negative term $\bigwedge_{i=1}^n \overline{x}_i$ for $w = \mathbf{0}$.

We first note simple relations between canonical Horn terms and co-Horn terms.

Proposition 4.2 *Let $t_v \in R(v)$ and $t_w \in$ co-$R(w)$ for $v \in T$ and $w \in F$. Then $T(t_v) \cap T(t_w) \neq \emptyset$ implies $v \leq w$.*

Proof. If $v \not\leq w$, then there is an $i \in P(t_v) \cap N(t_w)$, which clearly means $T(t_v) \cap T(t_w) = \emptyset$. □

Proposition 4.3 *Let $t_v \in R(v)$ and $t_w \in$ co-$R(w)$ for $v \in T$ and $w \in F$ with $v \leq w$. Then $T(t_v) \cap T(t_w) = \emptyset$ implies $N(t_v) = P(t_w)$.*

Proof. It is easy to see that $T(t_v) \cap T(t_w) = \emptyset$ holds if and only if $(P(t_v) \cap N(t_w)) \cup (N(t_v) \cap P(t_w)) \neq \emptyset$ holds. Then $v \leq w$ implies $ON(v) \cap OFF(w) = \emptyset$. Since $P(t_v) = ON(v)$ and $N(t_w) = OFF(w)$, we have $P(t_v) \cap N(t_w) = \emptyset$, and hence $N(t_v) \cap P(t_w) \neq \emptyset$. This means $N(t_v) = P(t_w)$, since $|N(t_v)| \leq 1$ and $|P(t_w)| \leq 1$. □

The following lemma describes the existence of a bidual Horn extension in terms of Horn and co-Horn functions.

Lemma 4.4 *A pdBf (T, F) has a bidual Horn extension f if and only if there exist a Horn function g_1 and a co-Horn function g_2 such that $T(g_1) \supseteq T$, $T(g_2) \supseteq F$ and $T(g_1) \cap T(g_2) = \emptyset$.*

Proof. Let us first show the only-if-part. Take $g_1 = f$ and $g_2 = \overline{g}_1$. Then $T(g_1) \supseteq T$, $T(g_2) \supseteq F$ and $T(g_1) \cap T(g_2) = \emptyset$ all hold. From the definition, obviously g_1 is Horn and g_2 is co-Horn.

To prove the if-part, let g_1 and g_2 be as stated in Lemma 4.4, and satisfy that $|T(g_1) \cup T(g_2)|$ is maximum. We show that $g_2 = \overline{g}_1$, which means that $g_1^d = (\overline{g}_1)^* = g_2^*$ is Horn, and hence shows that $f = g_1$ is a bidual Horn extension.

Towards a contradiction, assume that there is a vector $v \in \{0,1\}^n \setminus (T(g_1) \cup T(g_2))$. Then, $v \neq \mathbf{1}$, since otherwise $g_1' = g_1 \vee \bigwedge_{j=1}^n x_j$ is a Horn function satisfying $T(g_1') \cap T(g_2) = \emptyset$, which contradicts the maximality of $|T(g_1) \cup T(g_2)|$. Similarly, it is shown that $v \neq \mathbf{0}$.

Since $v \notin T(g_2)$, no canonical co-Horn term $t \in$ co-$R(v)$ is an implicant of g_2. Consequently, there is a vector $w \in T(g_1)$ such that $w \leq v$, since otherwise, Proposition 4.2 tells that any $t \in$ co-$R(v)$ satisfies $T(t_v) \cap T(t_w) = \emptyset$ for all $t_w \in R(w)$ with $w \in T(g_1)$. This implies $T(t_v) \cap T(g_1) = \emptyset$, which is a contradiction to the maximality of $|T(g_1) \cup T(g_2)|$. Since g_1 is Horn, some $t_1 \in R(w)$ is an implicant of g_1. Such a term t_1 satisfies that

- (i) $P(t_1) \subseteq ON(v)$ holds, since $P(t_1) = ON(w)$ and $ON(w) \subseteq ON(v)$.
- (ii) $N(t_1) \subseteq ON(v)$ holds, since otherwise (i.e., $N(t_1) \subseteq OFF(v)$) $v \in T(t_1)$ and hence $v \in T(g_1)$ would hold, in contradiction to the assumption.

Thus, by (i) and (ii), we have shown that g_1 has an implicant t_1 such that $V(t_1) \subseteq ON(v)$. Similarly, it is shown that g_2 has an implicant t_2 such that $V(t_2) \subseteq OFF(v)$.

Now g_1 and g_2 respectively have implicants t_1 and t_2 such that $V(t_1) \cap V(t_2) = \emptyset$. However, this clearly means that $T(t_1) \cap T(t_2) \neq \emptyset$, and hence $T(g_1) \cap T(g_2) \neq \emptyset$, which is a desired contradiction. This proves the if-part. □

Let us now define a bipartite graph $G_{(T,F)}$ on a pair of vertex sets $T' = T \setminus \{\mathbf{1}\}$ and $F' = F \setminus \{\mathbf{0}\}$ such that an edge is between $v \in T'$ and $w \in F'$ if $v \leq w$, and attach each vertex $v \in T'$ (resp., $w \in F'$) the set $I(v)$ (resp., co-$I(w)$) as label $L(v)$ (resp., $L(w)$). For a connected component C of $G_{(T,F)}$, let $L(C)$ be the intersection of all labels in C, i.e., $L(C) = \bigcap_{v \in C} L(v)$.

Lemma 4.5 *A pdBf (T, F) has a bidual Horn extension if and only if $L(C) \neq \emptyset$ for every connected component C of $G_{(T,F)}$.*

Proof. Note that g_1 and g_2 in Lemma 4.4 can be restricted to canonical Horn and co-Horn functions, respectively. Thus Lemma 4.4 and Propositions 4.2 and 4.3 imply this lemma (observe that $\mathbf{1} \not\leq w$, for every $w \in F$, and $v \not\leq \mathbf{0}$ for every $v \in T$). □

Now, we have the following algorithm.

Algorithm BH-EXTENSION
Input: A pdBf (T, F), where $T, F \subseteq \{0, 1\}^n$.
Output: "Yes", if there is a bidual Horn extension of (T, F); otherwise, "No".

Step 1. Construct the bipartite graph $G_{(T,F)}$.

Step 2. Compute all connected components C_i ($i = 1, 2, \ldots, k$) of $G_{(T,F)}$.

Step 3. **if** $L(C_i) \neq \emptyset$ holds for all i **then** output "Yes" **else** output "No" **fi**;
Halt. □

Theorem 4.6 *Given a pdBf (T, F), the existence of a bidual Horn extension of (T, F) can be checked in $O(n|T||F|)$ time.* □

Proof. The correctness of the algorithm follows from Lemma 4.5. Concerning the bound on the running time, Step 1 (i.e., constructing a bipartite graph $G_{(T,F)}$) can be executed in $O(n|T||F|)$ time. Step 2 can be done in $O(|T||F|)$ time by using a depth-first search, since the number of vertices and edges, respectively, are at most $|T| + |F|$ and $|T||F|$. Finally, Step 3 can be done in $O(n(|T| + |F|))$ time, since $L(C_i)$, $i = 1, 2, \ldots, k$, can be computed in $O(n(|T| + |F|))$ time. Totally, algorithm BH-EXTENSION requires $O(n|T||F|)$ time. □

Furthermore, the next lemma says that, if Step 3 is changed to the following Step 3′, then we can compute a bidual Horn extension of (T, F).

Step 3′. **if** $L(C_i) \neq \emptyset$ holds for all i **then**

$$\text{output} \qquad \varphi = \bigvee_{v \in T} t_v \vee \bigvee_{\substack{v,w \in T: \\ |N(t_v) \cup N(t_w)| = 2}} t^+_{v,w}, \tag{4.8}$$

where $t_v = \bigwedge_{j \in ON(v)} x_j \bar{x}_l$ for a fixed $l \in L(C_i)$ with $v \in C_i$, and $t_{\mathbf{1}} = x_1 x_2 \cdots x_n$
else output "No" **fi**;
Halt. □

Lemma 4.7 *For a pdBf* (T, F)*, let* $\varphi_1 = \bigvee_{t_j \in S} t_j$ *and* φ_2 *be Horn and co-Horn DNFs, respectively, satisfying* $T(\varphi_1) \supseteq T$*,* $T(\varphi_2) \supseteq F$*, and* $T(\varphi_1) \cap T(\varphi_2) = \emptyset$*. Then the DNF*

$$\varphi = \varphi_1 \vee \bigvee_{\substack{t_i, t_j \in S: \\ |N(t_i) \cup N(t_j)| = 2}} t^+_{i,j}$$

represents a bidual Horn extension of (T, F)*.* □

Note that $O(n|T|^2)$ time is needed, in order to output a bidual Horn DNF of (4.8) representing an extension of (T, F). Thus we have the following corollary.

Corollary 4.8 *Given a pdBf* (T, F)*, a bidual Horn extension of* (T, F) *can be computed in* $O(n|T|(|T| + |F|))$ *time (if any exists).* □

Example 4.3 Let us apply BH-EXTENSION to the pdBf (T, F) defined by $T = \{v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}\}$ and $F = \{w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}, w^{(5)}\}$, where $v^{(1)} = (0010)$, $v^{(2)} = (0100)$, $v^{(3)} = (0011)$, $v^{(4)} = (0101)$, $v^{(5)} = (0110)$, $w^{(1)} = (1001)$, $w^{(2)} = (1010)$, $w^{(3)} = (1100)$, $w^{(4)} = (1011)$, $w^{(5)} = (1101)$.

Step 1. The graph $G_{(T,F)}$ is shown in Figure 1:

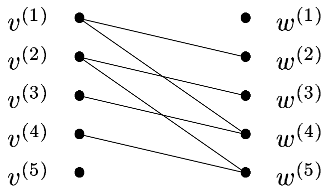

Figure 1: Graph $G_{(T,F)}$ of Example 5.5.

Step 2. The graph $G_{(T,F)}$ has four connected components, i.e., $C_1 = \{w^{(1)}\}$, $C_2 = \{w^{(2)}, v^{(1)}, w^{(4)}, v^{(3)}\}$, $C_3 = \{w^{(3)}, v^{(2)}, w^{(5)}, v^{(4)}\}$ and $C_4 = \{v^{(5)}\}$.

Step 3. We first compute the labels $L(u)$ of the vertices u by (4.5) and (4.7):

$L(v^{(1)}) = I(v^{(1)}) = \{1, 3\} \setminus \{3\} = \{1\}$ $\quad L(w^{(1)}) = \text{co-}I(w^{(1)}) = \{1, 2, 3, 4\} \setminus \{2, 3\} = \{1, 4\}$
$L(v^{(2)}) = I(v^{(2)}) = \{1, 2\} \setminus \{2\} = \{1\}$ $\quad L(w^{(2)}) = \text{co-}I(w^{(2)}) = \{1, 2, 4\} \setminus \{2, 4\} = \{1\}$
$L(v^{(3)}) = I(v^{(3)}) = \{1, 3, 4\} \setminus \{3, 4\} = \{1\}$ $\quad L(w^{(3)}) = \text{co-}I(w^{(3)}) = \{1, 3, 4\} \setminus \{3, 4\} = \{1\}$
$L(v^{(4)}) = I(v^{(4)}) = \{1, 2, 4\} \setminus \{2, 4\} = \{1\}$ $\quad L(w^{(4)}) = \text{co-}I(w^{(4)}) = \{1, 2\} \setminus \{2\} = \{1\}$
$L(v^{(5)}) = I(v^{(5)}) = \{1, 2, 3, 4\} \setminus \{2, 3\} = \{1, 4\}$ $\quad L(w^{(5)}) = \text{co-}I(w^{(5)}) = \{1, 3\} \setminus \{3\} = \{1\}$.

Thus, the connected components of G have labels $L(C_1) = \{1, 4\}$, $L(C_2) = \{1\}$, $L(C_3) = \{1\}$ and $L(C_4) = \{1, 4\}$. As a consequence, BH-EXTENSION outputs "Yes" (i.e., (T, F) has a bidual Horn extension).

Step 3'. By choosing $l = 1$ from $L(C_2)$, $L(C_3)$, and $L(C_4)$, we obtain a bidual Horn extension

$$\varphi_1 \;=\; x_3\overline{x}_1 \vee x_2\overline{x}_1 \vee x_3x_4\overline{x}_1 \vee x_2x_4\overline{x}_1 \vee x_2x_3\overline{x}_1 \;=\; x_3\overline{x}_1 \vee x_2\overline{x}_1$$

of (T, F), while by choosing $l = 1$ from $L(C_2)$, $L(C_3)$ and $l = 4$ from $L(C_4)$, we would obtain another bidual Horn extension

$$\begin{aligned}\varphi_2 &= x_3\overline{x}_1 \vee x_2\overline{x}_1 \vee x_3x_4\overline{x}_1 \vee x_2x_4\overline{x}_1 \vee x_2x_3\overline{x}_4 \vee x_2x_3 \vee x_2x_3 \vee x_2x_3x_4 \vee x_2x_3x_4 \\ &= x_3\overline{x}_1 \vee x_2\overline{x}_1 \vee x_2x_3.\end{aligned}$$ □

4.1 Computing all bidual Horn extensions

In this subsection, we briefly address the complexity of computing all bidual Horn extension of a pdBf (T, F). Since every positive function is a bidual extension of (T, F) if $T = F = \emptyset$, and there are clearly positive functions whose unique prime DNF is exponential in the number of variables, e.g., $\varphi = \bigvee_{S\subseteq\{1,2,\dots,n\}:|S|=\lfloor n/2\rfloor}(\bigwedge_{j\in S} x_j)$, a bidual Horn extension of (T, F) can require exponential space in the size of (T, F). Therefore, there is no algorithm for enumerating all bidual Horn extensions $\varphi_1, \varphi_2, \dots$ of (T, F) which spends only polynomial time on each extension φ_i.

It appears that even deciding, given a bidual Horn extension, whether an additional bidual Horn extension exists (rather than outputting one) is not easy. In fact, this problem is at least as hard as the positive duality problem (i.e., given positive DNFs φ and ψ, decide whether $f_\psi = f_\varphi^d$). The latter problem is is polynomially equivalent to a number of other problems, cf. [4, 11, 23, 15], but no polynomial time algorithm is known to date.

Theorem 4.9 *Let a pdBf (T, F) and a Horn DNF φ representing a bidual Horn extension of (T, F) be given. If checking whether there exists an extension $g \in \mathcal{C}_{BH}$ with $g \neq f_\varphi$ can be done in polynomial time, then the positive duality problem can be solved in polynomial time.*

Proof. (Sketch) We show this by a reduction of the Sperner saturation problem, which is known to be polynomially equivalent to the positive duality problem [11]: given a Sperner family $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ of subsets $S_i \subseteq V = \{1, 2, \dots, n\}$ (i.e., $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$ for all $i \neq j$), decide whether it is not saturated (i.e., there does exist a set $X \subseteq V$ such that $S_i \not\subseteq X \not\subseteq S_j$ holds for all i, j).

Given an instance of the Sperner saturation problem, we construct the instance of our problem as follows:

$$\begin{aligned} T &= \{x^{S_i} \mid i = 1, 2, \dots, m\} \cup \{x^{S_i\cup\{l\}} \mid i = 1, 2, \dots, m,\ \ l \in V \setminus S_i\} \\ F &= \{x^{S_i\setminus\{l\}} \mid i = 1, 2, \dots, m,\ \ l \in S_i\} \\ \varphi &= \bigvee_{i=1}^{m}(\bigwedge_{l\in S_i} x_l), \end{aligned}$$

where x^S is the characteristic vector of S, i.e., if $j \in S$, $x_j^S = 1$; otherwise, $x_j^S = 0$. Since φ is positive, it represents by Proposition 3.1 a bidual Horn extension of (T, F). Moreover, an extension $g \in \mathcal{C}_{BH}$ exists such that $g \neq f_\phi$ if and only if $\mathcal{S}$ is not saturated. □

Corollary 4.10 *Given a pdBf (T, F), deciding whether it has a unique extension $f \in \mathcal{C}_{BH}$ is at least as hard as the positive duality problem.* □

Corollary 4.11 *Let a pdBf (T, F) and a Horn DNF φ representing a bidual Horn extension of (T, F) be given. If checking whether there is a bidual Horn extension g such that $g \neq f_\varphi$ can be done in polynomial time, then there is a polynomial total time algorithm for the problem of dualizing a positive function, i.e, given a positive DNF, computing the irredundant prime DNF of its dual function.*

Proof. Follows from Theorem 4.9 and the result in [4] that a polynomial total time algorithm for dualizing a positive function exists if and only if the positive duality problem is polynomial. □

In the light of the open status of the positive duality problem, finding a polynomial time algorithm that decides whether another bidual Horn extension exists (similarly, whether a unique one exists) appears to be not straightforward.

5 Shortest Bidual Horn DNFs and Extensions

In this section, we consider the issue of shortest DNF representations of a bidual Horn function f, and of shortest bidual Horn extensions. More precisely, we consider the following problems: Compute (i) a term-shortest DNF, and (ii) a literal-shortest DNF, respectively, from a given Horn DNF of a bidual Horn function; similarly, compute (iii) a term-shortest bidual Horn extension, and (iv) a literal-shortest bidual Horn extension for a given pdBf (T, F).

A DNF φ is called *term-shortest* (resp., *literal-shortest*) if there is no DNF containing fewer terms (resp., a smaller total number of literals), which represents the same function f; a term-shortest (resp., literal-shortest) among the Horn DNFs that represent bidual Horn extensions of a pdBf (T, F) is called a *term-shortest* (resp. *literal-shortest*) *bidual Horn extension* of (T, F).

It is known [3, 27, 18] that problems (i) and (ii) for a general Horn function are both co-NP-hard. However, we shall show below that these problems for a bidual Horn function can be solved in polynomial time. Problems (iii) and (iv) turn out to be intractable.

5.1 Shortest DNFs for bidual Horn functions

Clearly, any term-shortest DNF is irredundant, and any literal-shortest DNF is irredundant and prime. Furthermore, there is a term-shortest DNF among prime ones. Thus we first describe some structural properties of irredundant prime DNFs of a general Horn function, which were proved in [19].

Since every prime implicant of a Horn function f is Horn, the set of all prime implicant of f is split into the set of all pure Horn prime implicants $Horn(f)$ and the set of all positive prime implicants $Pos(f)$. A function h is called the *pure Horn component* of a Horn function f if h satisfies the following conditions:

(i) h is pure Horn.

(ii) $f = h \vee g$, where g is a positive function.

(iii) If $f = h' \vee g'$, where h' and g' are pure Horn and positive, respectively, then $h' \geq h$.

The *pure Horn component* of a Horn function f is denoted by $h(f)$. The uniqueness of $h(f)$ follows directly from the definition. It was shown in [19] that $h(f)$ can be represented by a DNF $\bigvee_{t \in Horn(f)} t$; note that this DNF is not unique in general. This means that, given an arbitrary prime DNF of a Horn function f, $h(f)$ can be represented by the disjunction of all pure Horn terms therein. However, not much is known about the structure of term-shortest or literal-shortest DNFs that represent $h(f)$.

Contrary to this, the structure of the positive part g in the above (ii) is known to some extent. For this, we introduce a directed graph $G_P(f) = (Pos(f), A)$, where $A = \{(t_i, t_j) \mid t_j \leq h(f) \vee t_i\}$. It follows from the definition that $G_P(f)$ is transitively closed, and hence each strongly connected component of $G_P(f)$ is a complete directed subgraph. Let $Pos_i(f)$, $i = 1, 2, \ldots, k$, be the node sets of such strongly connected components of $G_P(f)$, among which $Pos_i(f)$, $i = 1, 2, \ldots l$, denote those having no incoming arcs. For a Horn function f, a positive DNF $\bigvee_{i=1}^{l} t_i$, where one $t_i \in Pos_i(f)$ is chosen for each of $i = 1, 2, \ldots, l$, is called a *positive restriction* of f. In general, different positive restrictions represent different positive functions. Obviously, the number of different positive restrictions is equal to $\prod_{i=1}^{l} |Pos_i(f)|$.

Example 5.1 Let f be a Horn function represented by a DNF:

$$\begin{aligned} \varphi^{(0)} = \; & \overline{x}_1 x_2 x_3 x_5 x_6 \vee x_1 x_3 x_4 \overline{x}_5 \vee x_1 x_2 x_4 \overline{x}_6 \vee \overline{x}_1 x_6 x_7 \\ & \vee x_1 x_2 x_3 x_4 \vee x_2 x_3 x_4 x_5 x_6 \vee x_1 x_2 x_4 x_7 \vee x_2 x_4 x_6 x_7 \vee x_3 x_6 x_7, \end{aligned}$$

where $\varphi^{(0)}$ is the disjunction of all prime implicants of f. Then

$$\begin{aligned} Horn(f) &= \{\overline{x}_1 x_2 x_3 x_5 x_6,\; x_1 x_3 x_4 \overline{x}_5,\; x_1 x_2 x_4 \overline{x}_6,\; \overline{x}_1 x_6 x_7\} \\ Pos(f) &= \{x_1 x_2 x_3 x_4,\; x_2 x_3 x_4 x_5 x_6,\; x_1 x_2 x_4 x_7,\; x_2 x_4 x_6 x_7,\; x_3 x_6 x_7\}. \end{aligned}$$

Note that this f is bidual Horn because the condition in Lemma 3.4 holds. Figure 2 shows the graph $G_P(f)$, and $Pos(f)$ can be divided into $Pos_1(f) = \{x_1x_2x_3x_4, x_2x_3x_4x_5x_6\}$, $Pos_2(f) = \{x_1x_2x_4x_7, x_2x_4x_6x_7\}$ and $Pos_3(f) = \{x_3x_6x_7\}$. This f satisfies $k = l = 3$. □

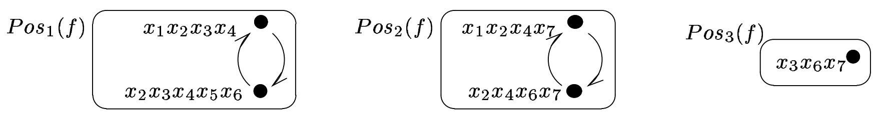

Figure 2: The graph $G_P(f)$ of f in Example 5.1.

The following lemma clarifies the role of positive restrictions.

Lemma 5.1 [19] *A DNF φ is an irredundant prime representation of a Horn function f if and only if*

$$\varphi = \varphi_H \vee \varphi_P,$$

where φ_H is an irredundant prime DNF of $h(f)$, and φ_P is a positive restriction of f. □

Let φ be an irredundant prime DNF of a Horn function f. Then, the above lemma tells that such φ always has the same number of positive terms. Therefore, a term-shortest DNF of a Horn function f can be obtained if we can find a term-shortest DNF for $h(f)$.

Example 5.2 Let us consider the Horn function f in Example 5.1. In this case, all pure Horn terms in $\varphi^{(0)}$ are irredundant. Then we have the following four irredundant prime DNFs of f.

$$\begin{aligned}
\varphi^{(1)} &= \overline{x}_1x_2x_3x_5x_6 \vee x_1x_3x_4\overline{x}_5 \vee x_1x_2x_4\overline{x}_6 \vee \overline{x}_1x_6x_7 \vee x_1x_2x_3x_4 \vee x_1x_2x_4x_7 \vee x_3x_6x_7,\\
\varphi^{(2)} &= \overline{x}_1x_2x_3x_5x_6 \vee x_1x_3x_4\overline{x}_5 \vee x_1x_2x_4\overline{x}_6 \vee \overline{x}_1x_6x_7 \vee x_1x_2x_3x_4 \vee x_2x_4x_6x_7 \vee x_3x_6x_7,\\
\varphi^{(3)} &= \overline{x}_1x_2x_3x_5x_6 \vee x_1x_3x_4\overline{x}_5 \vee x_1x_2x_4\overline{x}_6 \vee \overline{x}_1x_6x_7 \vee x_2x_3x_4x_5x_6 \vee x_1x_2x_4x_7 \vee x_3x_6x_7,\\
\varphi^{(4)} &= \overline{x}_1x_2x_3x_5x_6 \vee x_1x_3x_4\overline{x}_5 \vee x_1x_2x_4\overline{x}_6 \vee \overline{x}_1x_6x_7 \vee x_2x_3x_4x_5x_6 \vee x_2x_4x_6x_7 \vee x_3x_6x_7. \quad \Box
\end{aligned}$$

Now let us restrict our attention to bidual Horn functions.

Lemma 5.2 *Let f be a bidual Horn function. Then all pure Horn prime implicants of f are essential* (*hence irredundant*).

Proof. Let $t_1 = (\bigwedge_{j\in P(t_1)} x_j)\overline{x}_k$ be a pure Horn prime implicant of f, and let v be the vector defined by $ON(v) = P(t_1)$. Obviously, $t_1(v) = 1$, and hence $f(v) = 1$. We claim that t_1 is the unique prime implicant of f such that $t_1(v) = 1$, which completes the proof. For this, let us assume that t_2 $(\neq t_1)$ is a prime implicant of f satisfying $t_2(v) = 1$, that is,

$$P(t_2) \subseteq ON(v)\,(= P(t_1)). \tag{5.9}$$

Since all prime implicants of a Horn function are Horn, t_2 is either positive or pure Horn. However, if t_2 is positive, then $t_2 > t_1$ holds by (5.9), implying that t_1 is not a prime implicant of f, which is a contradiction. Therefore, t_2 is pure Horn, but satisfies $N(t_2) \neq \{k\}$, since otherwise t_1 is not a prime implicant of f by (5.9). Then $t^+_{1,2} \leq f$ holds by Lemma 3.3. However, by (5.9), this again implies that t_1 is not a prime implicant of f, a contradiction. $\Box$

This explains why all pure Horn terms in $\varphi^{(0)}$ of Example 5.1 are irredundant. Consequently, any irredundant prime DNF of a bidual Horn function f can be represented by $\varphi = \bigvee_{t\in Horn(f)} t \vee \varphi_P$, where φ_P is a positive restriction of f. This means that, if we restrict our attention to prime DNFs, the class of irredundant DNFs of f is the same as the class of term-shortest DNFs of f.

Lemma 5.3 *Let f be a bidual Horn function. Then φ is an irredundant prime DNF of f if and only if φ is a term-shortest prime DNF of f.* $\Box$

Note that this lemma does not imply the uniqueness of the term-shorest prime DNF of a bidual function. Indeed, the bidual Horn function f in Example 5.1 has four term-shortest prime DNFs given in Example 5.2. Let φ be any Horn DNF of a Horn function f. Since an irredundant prime DNF of f can be computed from φ in $O(|\varphi|^2)$ time [19], we have the following theorem.

Theorem 5.4 *Let φ is a Horn DNF of a bidual Horn function f. Then a term-shortest prime DNF of f can be computed from φ in $O(|\varphi|^2)$ time.* □

Let us next turn our attention to the literal-shortest DNFs of bidual Horn functions. We can see that some term-shortest DNFs are not literal-shortest; e.g., $\varphi^{(3)}$ and $\varphi^{(4)}$ in Example 5.2 are term-shortest but not literal-shortest. Lemma 5.1 and Lemma 5.2 imply the following.

Lemma 5.5 *Let f be a bidual Horn function. Then φ is a literal-shortest DNF of f if and only if*

$$\varphi = \bigvee_{t \in Horn(f)} t \ \vee \ \varphi_P,$$

where φ_P is a literal-shortest positive restriction of f. □

For such a φ_P, we must find a $t' \in Pos_i(f)$ having the minimum $|t'|$ for each $Pos_i(f)$ with $i \leq l$ (recall that such $Pos_i(f)$ corresponds to a connected component of $G_P(f)$ having no incoming arc).

Lemma 5.6 *Let f be a Horn function, and let $t \in Pos_i(f)$ with $i \leq l$. Then a positive term t' satisfies $t' \in Pos_i(f)$ if and only if $t' \in Pos(f)$ and $t \leq h(f) \vee t'$ (i.e., $G_P(f)$ has the arc (t', t)).*

Proof. The only-if part is obvious. To prove the if-part, it is sufficient to show $t' \leq h(f) \vee t$ (i.e., $G_P(f)$ has arc (t, t')). Let us assume the contrary. Then $Pos_i(f)$ has an incoming arc from $Pos_j(f)$ that contains t' (i.e., $i > l$), which is a contradiction. □

To develop an efficient algorithm to compute a short t', we need several elementary properties of Horn consensus procedures.

Lemma 5.7 *Let t_1, t_2 and t_3 be pure Horn terms. If t_3 is the consensus of an ordered pair of terms t_1 and t_2, then t_1, t_2, and t_3 satisfy*

(i) $N(t_3) = N(t_1)$ *and* $N(t_2) \cap V(t_3) = \emptyset$.

(ii) $P(t_3) = P(t_2) \cup (P(t_1) \setminus N(t_2))$ *and* $N(t_2) \subseteq P(t_1)$. □

Lemma 5.8 *Let a positive term t_3 be the consensus of an ordered pair of terms t_1 and t_2. Then t_1, t_2, and t_3 satisfy*

(i) $N(t_1) = \emptyset$ *and* $|N(t_2)| = 1$ *(i.e., t_1 is positive and t_2 is pure Horn).*

(ii) $P(t_3) = P(t_2) \cup (P(t_1) \setminus N(t_2))$ *and* $N(t_2) \subseteq P(t_1)$.

(iii) $N(t_2) \cap V(t_3) = \emptyset$. □

Recall that, in the above lemmas, t_1 and t_2 are, according to the usual terminology, the left-parent and the right-parent of t_3, respectively, and t_3 is the child of t_1 as well as being the child of t_2. Let $child(t)$ denote the child of t.

Now, given a Horn DNF φ of a Horn function f, assume that there is a prime implicant t of f, which is not in φ. Then there is a sequence

$$L = t^{(1)}, t^{(2)}, \ldots, t^{(m)} \; (= t)$$

such that each $t^{(k)}$ is either in φ or the consensus of two terms $t^{(k_1)}$ and $t^{(k_2)}$ with $k_1, k_2 < k$. We call t' a *right-ancestor* (resp., *left-ancestor*) of t with respect to L if either t' is the right-parent (resp., left-parent) of t, or t' is a right-ancestor (resp., left-ancestor) of the right-parent (resp., left-parent) of t. Furthermore, we call a right-ancestor (resp., left-ancestor) t' of t the *right-root* (resp., *left-root*) of t if t' is in φ. By definition, the right-root (resp., left-root) of t is unique. For example, for $\varphi = x_1\overline{x}_2 \vee x_2\overline{x}_3 \vee x_3x_4\overline{x}_5$, consider a consensus sequence $L = x_1\overline{x}_2$, $x_2\overline{x}_3$, $x_3x_4\overline{x}_5$, $x_1\overline{x}_3$, $x_1x_4\overline{x}_5$ leading to $x_1x_4\overline{x}_5$ (see Figure 3). Then $x_1\overline{x}_3$ is the right-parent, $x_3x_4\overline{x}_5$ is the left-parent and left-root, and $x_1\overline{x}_2$ is the right-root of $x_1x_4\overline{x}_5$, respectively.

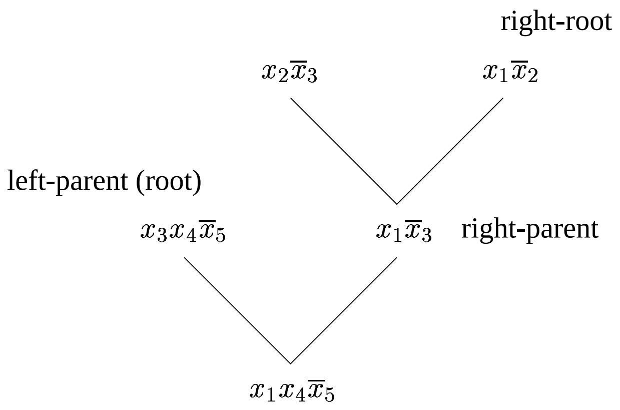

Figure 3: A consensus sequence leading to $x_1x_4\overline{x}_5$ for a Horn DNF $x_1\overline{x}_2 \vee x_2\overline{x}_3 \vee x_3x_4\overline{x}_5$.

Lemma 5.9 *Let φ be a Horn DNF of a Horn function f. For a prime implicant t of f which is not in φ, let L be a consensus sequence leading to t. Then every right-ancestor t^* of t satisfies $P(t^*) \subseteq P(t)$.* $\square$

Let f be a bidual Horn function, and let t_i and t_j be Horn implicants of f such that $|N(t_i) \cup N(t_j)| = 2$. Let $t^s_{i,j}$ denote a positive term of minimum length, which satisfies $t^+_{i,j} \le t^s_{i,j} \le t_i \vee t_j \vee t^+_{i,j}$. By Lemma 3.3 (ii), $t^s_{i,j}$ is an implicant of f. It is not difficult to see that $t^s_{i,j}$ can be obtained from $t^+_{i,j}$ by deleting a positive literal x_k if k satisfies $k \in N(t_i) \cap P(t_j)$ or $k \in N(t_j) \cap P(t_i)$, and $t^s_{i,j} = t^+_{i,j}$ if there is no such k. Therefore, there are at most two such $t^s_{i,j}$. For example, suppose $t_i = x_1x_2\overline{x}_3$ and $t_j = x_1\overline{x}_2x_3x_4$. Then $t^+_{i,j} = x_1x_2x_3x_4$, and hence $t^s_{i,j} = x_1x_2x_4$ or $x_1x_3x_4$.

Lemma 5.10 *Let f be a bidual Horn function, and assume that $t, t' \in Pos(f)$ satisfy $t \ne t'$ and $t' \le h(f) \vee t$ (i.e., $G_P(f)$ has the arc (t', t)). Then, either*

(*i*) *t' is the consensus of t and some $t_i \in Horn(f)$ (i.e., $t' = \bigwedge_{j \in P(t_i) \cup (P(t) \setminus N(t_i))} x_j$), or* (*ii*) $t' = t^s_{i,j}$ *for a term* $t^s_{i,j}$ *of some* $t_i, t_j \in Horn(f)$.

Proof. Clearly, t' is a prime implicant of $h(f) \vee t$. Hence, there is a sequence $t^{(1)}, t^{(2)}, \ldots, t^{(m)}\ (= t')$ such that each $t^{(k)}$ is either in $Horn(f) \cup \{t\}$ or the consensus of two terms $t^{(k_1)}$ and $t^{(k_2)}$ with $k_1, k_2 < k$. Since $t \neq t'$, $m \geq 3$ must hold. By Lemma 5.8 (i), $t'\ (= t^{(m)})$ is the consensus of a positive term $t^{(m_1)}$ and a pure Horn term $t^{(m_2)}$ such that $m_1, m_2 < m$. If $m = 3$ (i.e., $t^{(m_1)} = t$ and $t^{(m_2)} \in Horn(f)$), then obviously (i) holds. Otherwise (i.e., $m > 3$), the following two cases are possible.

Case (a): $t^{(m_2)} \notin Horn(f)$. Let t_2 be the right-root of t', and let t_1 be the left-parent of $t_3 = child(t_2)$ (see Figure 4). Since $h(f) \vee t$ is Horn, only t' and its left-ancestors are positive (i.e., all other terms in the sequence are pure Horn). Thus, t_1, $child(t_1)$ and t_2 are pure Horn. Then by Lemma 5.7, we have

$$N(t_1) \neq N(t_2) \tag{5.10}$$
$$P(t_1) \subseteq P(t_3) \cup N(t_2) \tag{5.11}$$
$$P(t_2) \subseteq P(t_3). \tag{5.12}$$

Since t_3 is a right-ancestor of t', Lemma 5.9 tells that $P(t_3) \subseteq P(t')$; hence, (5.11) and (5.12) become

$$P(t_1) \subseteq P(t') \cup N(t_2) \tag{5.13}$$
$$P(t_2) \subseteq P(t'). \tag{5.14}$$

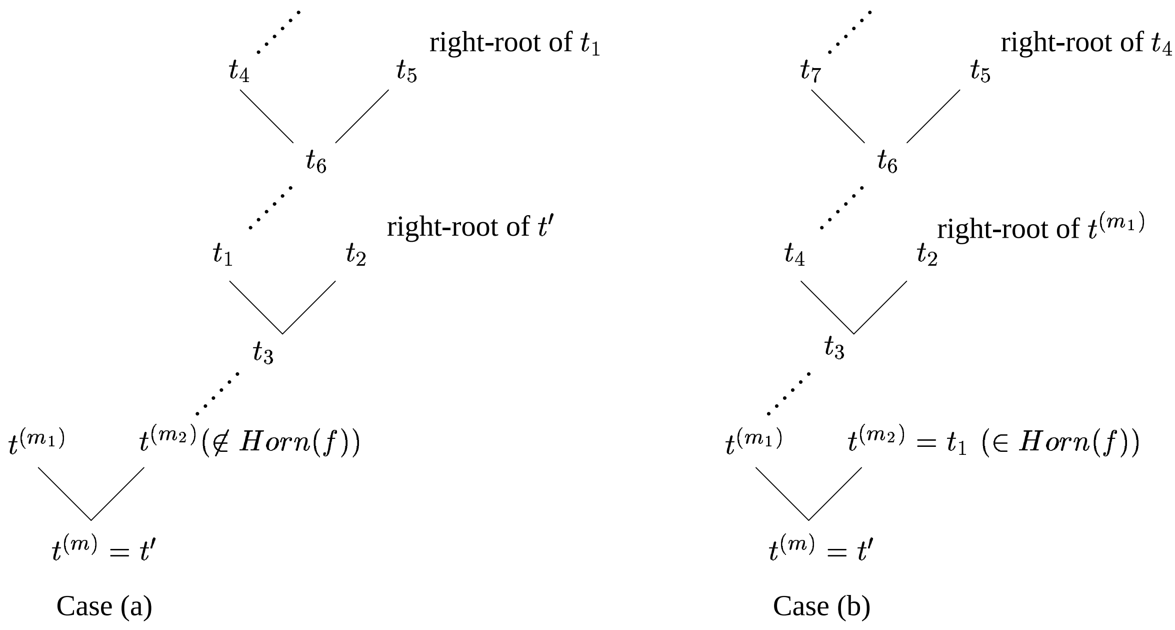

Figure 4: The cases (a) and (b) in the proof of Lemma 5.11.

Then, under condition (5.10), we have $t' \leq t^s_{1,2}$ from (5.13) and (5.14), because $t^s_{1,2}$ does not contain the positive literal x_k with $\{k\} = N(t_2)$ in either case of $N(t_2) \subseteq P(t_1)$ or $N(t_2) \cap P(t_1) = \emptyset$. Since t_1, t_2 are Horn implicants of f and f is bidual, $t^2_{1,2}$ is an implicant of f. Since t' is a prime implicant of f, it follows $t' = t^s_{1,2}$. Hence, clause (ii) holds if $t_1 \in Horn(f)$. Tho show the latter, assume that $t_1 \notin Horn(f)$. Let t_5 be the right-root of t_1, and t_4 be the left-parent of $t_6 = child(t_5)$ (see Figure 4). From the above observation on positive and negative terms in the consensus sequence, t_5 is pure Horn. Therefore, by Lemma 5.9 and (5.13), we have

$$P(t_5) \subseteq P(t_1) \subseteq P(t') \cup N(t_2).$$

Consider the negative literals of t_5 and t_2. If $N(t_5) \neq N(t_2)$, then by (5.14) it follows that a $t^s_{5,2}$ exists with $t' \leq t^s_{5,2}$ (if $N(t_2) = \{k\} \subseteq P(t_2)$, then remove x_k from $t^+_{5,2}$). Since t_2, t_5 are Horn implicants of f and f is bidual, $t^s_{5,2}$ is an implicant of f. Since t' is a prime implicant of f, it follows $t' = t^s_{5,2}$. Hence, clause (ii) is true. On the other hand, if $N(t_5) = N(t_2)$, then t_4

$$N(t_4) \neq N(t_5) = N(t_2) \tag{5.15}$$

holds by (i) of Lemma 5.7 (recall that t_4 must be pure Horn). Furthermore,

$$\begin{aligned} P(t_4) &\subseteq P(t_6) \cup N(t_5) \\ &\subseteq P(t_1) \cup N(t_2) \quad \text{(by Lemma 5.9 and } N(t_5) = N(t_2)) \\ &\subseteq P(t') \cup N(t_2) \quad \text{(by (5.13))}. \end{aligned} \tag{5.16}$$

Along the same line of argumentation as in the case $N(t_5) \neq N(t_2)$, we obtain from (5.14)–(5.16) that $t' = t^s_{2,4}$ for some term $t^s_{2,4}$, which implies that clause (ii) holds if $t_4 \in Horn(f)$. If $t_4 \notin Horn(f)$, we can repeat for t_4 the same argument as for $t_1 \notin Horn(f)$ by considering the right-root t_8 of t_4 and the left parent t_7 of $t_9 = child(t_8)$ and so on. Since the consensus sequence is finite, this can not be repeated indefinitely, and we must eventually encounter some $t_i \in Horn(f)$ such that $t^s_{i,2} = t'$ and clause (ii) holds. This completes the proof of Case (a).

Case (b): $t^{(m_2)} \in Horn(f)$ *(see Figure 4)*. Let $t_1 = t^{(m_2)}$ and let t_2 be the right-root of $t^{(m_1)}$ (since $m > 3$ we have $t^{(m_1)} \notin Horn(f) \cup \{t\}$). Note that t_1 and t_2 are pure Horn. It follows from Lemmas 5.8 and 5.9 that

$$\begin{aligned} P(t_1) &\subseteq P(t') \\ P(t_2) &\subseteq P(t') \cup N(t_1). \end{aligned}$$

If $N(t_1) \neq N(t_2)$, then along the same line of argumentation as in Case (a), we obtain $t' = t^s_{1,2}$ for some term $t^s_{1,2}$. Since $t_2 \in Horn(f)$, clause (ii) is satisfied.

Otherwise (i.e., $N(t_2) = N(t_1)$), consider the left-parent t_4 of $t_3 = child(t_2)$ (see Figure 4.(b)). Note that t_2 can not be the right-parent of $t^{(m_1)}$; for, then $t_4 = t$ is the left-parent of $t^{(m_1)}$, and $N(t_2) \not\subseteq P(t^{(m_1)})$, which by $N(t_1) = N(t_2)$ implies that the pair $t^{(m_1)}, t^{(m_2)}$ has no consensus. Consequently, all terms in the tree with root t_3 ($= child(t_2)$), and in particular t_4, are pure Horn.

We have $N(t_4) \neq N(t_2)$ and

$$\begin{array}{rcll} P(t_4) & \subseteq & P(t_5) \cup N(t_2) & \text{(Lemma (5.7))} \\ & \subseteq & P(t^{(m_1)}) \cup N(t_2) & \text{(Lemma (5.9))} \\ & \subseteq & P(t') \cup N(t_1) & (N(t_2) = N(t_1)) \end{array}$$

On the other hand, we have already shown

$$P(t_1) \subseteq P(t').$$

Hence, there is some $t^s_{1,4}$ such that $t' \geq t^s_{1,4}$. Since f is bidual and t_1, t_4 are Horn implicants of f, it follows that $t = t^s_{1,4}$. Thus, if $t_4 \in Horn(f)$, then clause (ii) holds. If $t_4 \notin Horn(f)$, then we repeat the argument by considering the right-root t_5 of t_4, the left parent t_7 of $t_6 = child(t_5)$ and so on. Since the consensus sequence is finite, eventually we encounter a term $t_i \in Horn(f)$ such that $t^s_{1,i} = t'$; hence, clause (ii) holds. This completes the proof. □

The previous lemma says that, for any arc (t, t') in $G_P(f)$, t' satisfies either (i) or (ii). The next lemma sharpens this by stating that clause (ii) must hold if t' is shorter than t.

Lemma 5.11 *Let f be a bidual Horn function, and let $t, t' \in Pos(f)$ satisfy $t \neq t'$ and $t' \leq h(f) \vee t$. If $|t'| < |t|$ holds, then t' satisfies the case* (ii) *of Lemma* 5.10.

Proof. Assume that t' is the consensus of t and some $t_i \in Horn(f)$ (i.e., $t' = \bigwedge_{j \in P(t_i) \cup (P(t) \setminus N(t_i))} x_j$). Since $|t'| < |t|$, $P(t') = P(t) \setminus N(t_i)$ must hold. This means that t is not a prime implicant of f, which is a contradiction. □

Now, we describe an algorithm L-SHORTEST to compute a literal-shortest DNF of a bidual Horn function.

Algorithm L-SHORTEST

Input: A Horn DNF φ of a bidual Horn function f.

Output: A literal-shortest DNF of f.

Step 1. Compute an irredundant prime Horn DNF φ' of f from φ.

Step 2. Let φ'_H (resp., φ'_P) be the disjunction of all Horn (resp., positive) terms in φ';
for each t in φ'_P **do**
Find a term $t^s_{i,j}$ with the minimum $|t^s_{i,j}|$ among those satisfying $t \leq \varphi'_H \vee t^s_{i,j}$, where t_i and t_j are in φ'_H and satisfy $|N(t_i) \cup N(t_j)| = 2$;
if such a $t^s_{i,j}$ exists and $|t^s_{i,j}| < |t|$ **then** replace t in φ'_P by $t^s_{i,j}$ **fi**
end.

Step 3. Output DNF $\varphi' = \varphi'_H \vee \varphi'_P$ and halt. □

Theorem 5.12 *Given a Horn DNF φ of a bidual Horn function f, algorithm* L-SHORTEST *correctly outputs a literal-shortest DNF of f in $O(|\varphi|(m_h^2 m_p + |\varphi|))$ time, where m_h and m_p denote the numbers of Horn and positive terms in φ, respectively.*

Proof. Let us first show the correctness of algorithm L-SHORTEST. In Step 1, it computes an irredundant prime DNF of f. Since φ'_H is equal to $\bigvee_{t \in Horn(f)} t$ by Lemmas 5.2 and 5.5, φ'_H represents the pure Horn component $h(f)$ of f.

Let us next consider the positive restriction φ'_P of f. In Step 2, each t in φ'_P is replaced by some shortest $t^s_{i,j}$ among those $t^s_{i,j}$ which satisfy $t \leq \varphi'_H \vee t^s_{i,j}$ (provided any such $t^s_{i,j}$ exists); let this term be $t^{ss}_{i,j}$. Let $t \in Pos_q(f)$, where $q \leq l$ (l was defined before Example 5.1). Then, by Lemma 5.11, it is sufficient to show that such $t^{ss}_{i,j}$ satisfies $t^{ss}_{i,j} \in Pos_q(f)$. Note that $t^{ss}_{i,j}$ is an implicant of f since f is bidual Horn. By Lemma 5.6, obviously $t^{ss}_{i,j} \in Pos_q(f)$ holds if $t^s_{i,j} \in Pos(f)$. To complete the proof of the correctness part, it thus remains to prove $t^{ss}_{i,j} \in Pos(f)$. Assume the contrary. As a consequence, there exists a $t' \in Pos(f)$ such that $t' > t^{ss}_{i,j}$. This t' satisfies $t \leq \varphi'_H \vee t'$, and hence, by Lemma 5.6, $t' \in Pos_q(f)$. Since $|t'| < |t^{ss}_{i,j}| < |t|$, Lemma 5.11 tells that t' is obtained by clause (ii) there. It follows that $t^{ss}_{i,j}$ is not shortest among the $t^s_{i,j}$, which satisfy $t \leq \varphi'_H \vee t^s_{i,j}$. We arrived at a contradiction. This proves the correctness of L-SHORTEST.

Finally, let us consider the time complexity of algorithm L-SHORTEST. Step 1 can be done in $O(|\varphi|^2)$ time [19]. In Step 2, for each pair of t_i and t_j, we can construct the terms $t^s_{i,j}$ in $O(n)$ time. Each test $t \leq \varphi'_H \vee t^s_{i,j}$ can be done in $O(|\varphi|)$ time. Since $n \leq |\varphi|$, Step 2 requires $O(m_h^2 m_p |\varphi|)$ time in total. Step 3 can be executed in $O(|\varphi'|) = O(|\varphi|)$ time. Consequently, algorithm L-SHORTEST can be executed in $O(|\varphi|(m_h^2 m_p + |\varphi|))$ time. □

Example 5.3 Let us consider the bidual Horn function f of Example 5.1. Suppose that DNF $\varphi^{(3)}$ in Example 5.2 is an input (i.e., $\varphi := \varphi^{(3)}$). Since $\varphi^{(3)}$ is irredundant prime, $\varphi' = \varphi\, (= \varphi^{(3)})$ holds in Step 1. Thus

$$\begin{aligned} \varphi'_H &= \overline{x}_1 x_2 x_3 x_5 x_6 \vee x_1 x_3 x_4 \overline{x}_5 \vee x_1 x_2 x_4 \overline{x}_6 \vee \overline{x}_1 x_6 x_7, \quad \text{and} \\ \varphi'_P &= x_2 x_3 x_4 x_5 x_6 \vee x_1 x_2 x_4 x_7 \vee x_3 x_6 x_7. \end{aligned}$$

In Step 2, for $t = x_2 x_3 x_4 x_5 x_6$ in φ'_P, we find that $t^s_{1,2} = x_1 x_2 x_3 x_4$ obtained from $t_1 = x_1 x_3 x_4 \overline{x}_5$ and $t_2 = x_1 x_2 x_4 \overline{x}_6$ has the minimum $|t^s_{1,2}|$ among those satisfying $t \leq \varphi'_H \vee t^s_{1,2}$. Since this $t^s_{1,2}$ satisfies $|t^s_{1,2}| < |t|$, t is replaced by $t^s_{1,2}$. For the other terms t in φ'_P, there is no $t^s_{i,j}$ having the desired property. Thus in Step 3, algorithm L-SHORTEST outputs $\varphi^{(1)}$ of Example 5.2, which is in fact a literal-shortest DNF of f. □

The previous theorem, together with the results about recognition of bidual Horn functions in Section 3, imply that, given a Horn DNF φ, we can recognize in polynomial time whether φ represents a bidual Horn function and, if this is the case, compute in polynomial time a literal-shortest or term-shortest prime DNF of f. Thus, bidual Horn functions constitute a polynomial-time recognizable subclass of Horn functions for which computing a literal-shortest and term-shortest prime DNF is polynomial.

By Lemmas 5.2 and 5.10, we have an interesting property regarding the number of prime implicants of a bidual Horn function.

Theorem 5.13 *Let φ be an irredundant prime DNF of a bidual Horn function f, and let m_h and m_p denote the numbers of Horn and positive terms in φ, respectively. Then:*

(a) $|Horn(f)| = m_h$, *and*

(b) $m_p \leq |Pos(f)| \leq 2m_h^2 + m_p(m_h + 1)$.

Proof. (a) is immediate from Lemma 5.2, and the disjunction of all pure Horn terms in φ gives the pure Horn component $h(f)$ of f. For (b), $m_p \leq |Pos(f)|$ is obvious from the definition of $Pos(f)$. Furthermore, every $t' \in Pos(f)$ satisfies $t' \leq h(f) \vee t$ for some positive term t in φ because, by Lemma 5.1, the disjunction of all positive terms in φ represents the positive restriction φ_P of f. All such $t' \in Pos(f)$ are obtained either by (i) or by (ii) of Lemma 5.10. There are at most $m_p m_h$ t' of type (i), and at most $2m_h^2$ t' of type (ii). Totally, $|Pos(f)| \leq m_p + m_p m_h + 2m_h^2 = 2m_h^2 + m_p(m_h + 1)$ holds. □

5.2 Shortest bidual Horn extensions

Algorithm BH-EXTENSION from Section 4 allows to compute a Horn DNF representing a bidual Horn extension of (T, F) in polynomial time, but Step 3$'$ might lead to a quite large expression (of quadratic size) in the worst case. This step an be refined such that a DNF smaller than φ is computed in polynomial time; in fact, as follows by the results in Section 5.1, it is possible to compute even a term-shortest or literal-shortest DNF equivalent to φ in polynomial time.

A natural issue is whether a term-shortest resp. literal-shortest bidual Horn extension of (T, F), can be computed in polynomial time. These problems are intractable, however.

Theorem 5.14 *Given a pdBf (T, F), computing (i) a term-shortest, or a (ii) literal-shortest bidual Horn φ of (T, F) is* NP-*hard.*

Proof. Part (i) follows from the reduction in the proof of [6, Theorem 9] which shows that deciding whether a pdBf (T, F) has an extension represented by a DNF of at most h terms, is NP-hard; this holds even if h is fixed to 2. With a given 3-uniform hypergraph $\mathcal{H} = (V, E)$, i.e., E is a collection of 3-element subsets of a finite set V, associate a pdBf (T, F), $T, F \subseteq \{0, 1\}^{|V|}$ by defining

$$T = \{x^{V \setminus \{j\}} \mid j \in V\}, \quad F = \{x^{V \setminus H} \mid H \in E\}.$$

As shown in [6], the pdBf (T, F) has an extension with DNF $\varphi = t_1 \vee t_2$ if and only if the hypergraph $\mathcal{H}$ is 2-colorable. (Note that if (C_1, C_2) is a good 2-coloring of $\mathcal{H}$, then $\varphi = \bigwedge_{i \in C_1} x_i \vee \bigwedge_{j \in C_2} x_j$ is an extension of (T, F).) Moreover, it is shown that among the term-shortest extensions of (T, F), i.e., extensions with smallest number of terms in a DNF, there is a positive function.

Since every positive function is bidual Horn, it follows that some term-shortest extension of (T, F) is bidual Horn. Furthermore, deciding 2-colorability of a 3-uniform hypergraph $\mathcal{H}$ is a well-known NP-complete problem (cf. [17]). As a consequence, computing a term-shortest bidual Horn extension of (T, F) is NP-hard.

For part (ii), we reduce the classical NP-hard problem of deciding whether a graph $G = (V, E)$ has a vertex cover of size at most k [17] to this problem. Suppose that $V = \{1, 2, \ldots, n\}$, and define $T = \{\mathbf{1}\}$ and $F = \{x^{V \setminus \{i,j\}} \mid \{i, j\} \in E\}$. Then we claim that (T, F) has a bidual Horn DNF φ containing at most k literals if and only if G has a vertex cover of size at most k. Indeed, if C is a vertex cover of G, then $\varphi = \bigwedge_{i \in C} x_i$ represents an extension, which is clearly bidual Horn. This proves the if-part. To show the only-if-part, assume that G has no vertex cover of size at most k, and let φ represent any Horn (in particular, bidual Horn) extension. Then $t(\mathbf{1}) = 1$ holds for some Horn implicant t of φ. Let $t = \bigwedge_{j \in P(t)} x_j \bigwedge_{j \in N(t)} \overline{x}_j$. Since $N(t) = \emptyset$ must hold, $P(t)$ is a vertex cover of G. Hence, φ contains at least $|P(t)| \geq k + 1$ literals, a contradiction. □

Remark. The proof of Theorem 5.14 implies that computing a literal-shortest arbitrary (not necessarily Horn DNF) formula φ representing any bidual Horn extension is NP-hard as well.

6 Dualization, Characteristic Sets, and DNFs of Bidual Horn Functions

In what follows, let $\circ$ denote either conjunction $\wedge$ or disjunction $\vee$. Let $S \subseteq \{0, 1\}^n$ be a set of vectors such that $S = Cl_\circ(S)$, i.e., S is closed under intersection or union, respectively. A vector $v \in S$ is called $\circ$-*extreme* with respect to S if $v \notin Cl_\circ(S \setminus \{v\})$. The set of all $\circ$-extreme vectors of S is called the $\circ$-*characteristic set* of S, which we denote by $C^*_\circ(S)$; the definition extends to all $S \subseteq \{0, 1\}^n$ by $C^*_\circ(S) = C^*_\circ(Cl_\circ(S))$. Note that $C^*_\circ(X)$ is well-defined and is the minimum set satisfying $Cl_\circ(C^*_\circ(X)) = Cl_\circ(X)$.

For conjunction, the concept of characteristic set has been studied e.g. in [25, 24], and is also known as *base* [10]. The translation between the characteristic set and DNFs of Horn functions is an important problem which has been studied repeatedly [25, 24, 26]. Briefly, it appeared that a good algorithm for this problem is not straightforward and the intrinsic difficulty of this task is not known to date. In this section, we study the characteristic set of bidual Horn functions and consider major transformation problems on them. These results also allow us to characterize the difficulty of dualizing a bidual Horn function f, i.e., given a Horn DNF for f, compute an irredundant prime (Horn) DNF for f^d.

Let for any $S \subseteq \{0, 1\}^n$ denote $S^d = \{\overline{v} \mid v \in S\}$. Then obviously $C^*_\wedge(S) = (C^*_\vee(S^d))^d$ holds; therefore, we restrict our discussion to $C^*_\vee$.

For convenience, we introduce further notation. For any term t, let $T_-(t) = \{v \in T(t) \mid |ON(v)| \leq |P(t)| + 1\}$ and, furthermore, for a DNF $\varphi = \vee_{i \in I} t_i$, let $T_-(\varphi) = \{v \in T_-(t_i) \mid i \in I\}$.

Example 6.1 *Consider the DNF* $\varphi = x_1 x_2 \overline{x}_3 \vee \overline{x}_1 \overline{x}_4$ *on variables* $x_1, \ldots, x_4$. *Then,* $T_-(x_1 x_2 \overline{x}_3) = \{1100, 1101\}$, $T_-(\overline{x}_1 \overline{x}_4) = \{0000, 0100, 0010\}$, *and* $T_-(\varphi) = \{1100, 1101, 0000, 0100, 0010\}$.

Lemma 6.1 *For any term t, it holds that* $C^*_\vee(T(t)) = T_-(t)$.

Proof. First, observe that every $v \in T_-(t)$ is $\vee$-extreme with respect to $T(t)$, that is, $C^*_\vee(T(t)) \supseteq T_-(t)$. To prove $C^*_\vee(T(t)) \subseteq T_-(t)$, take a vector $u \in T(t) \setminus T_-(t)$. Then, for every $j \in ON(u) \setminus P(t)$, there is a $v^{(j)} \in T_-(t)$ such that $v^{(j)}_j = 1$, and thus we have

$$u = \bigvee \{v^{(j)} \mid j \in ON(u) \setminus P(t)\};$$

it follows $C^*_\vee(T(t)) \subseteq T_-(t)$. This proves the lemma. □

Lemma 6.2 *Let f be a function such that f^d is Horn (i.e., $T(f)$ is closed under union). Let φ be an arbitrary DNF of f. Then $C^*_\vee(T(f)) = C^*_\vee(T_-(\varphi))$.*

Proof. For every $u \in T(f) \setminus T_-(\varphi)$, there is a term $t \in \varphi$ such that $t(u) = 1$. Lemma 6.1 then implies $u \in Cl_\vee(T_-(t))$. Hence $Cl_\vee(T(f)) = T(f) \subseteq Cl_\vee(T_-(\varphi))$ holds. On the other hand, $T(f) \supseteq T_-(\varphi)$, hence $Cl_\vee(T(f)) \supseteq Cl_\vee(T_-(\varphi))$. It follows $Cl_\vee(T(f)) = Cl_\vee(T_-(\varphi))$, which implies $C^*_\vee(T(f)) = C^*_\vee(T_-(\varphi))$. □

By the above lemma, we can transform an arbitrary DNF of a function f such that f^d is Horn into the characteristic set $C^*_\vee(T(f))$ in polynomial time.

Theorem 6.3 *Let f be a function of n variables such that f^d is Horn (i.e., $T(f)$ is closed under union). Let φ be an arbitrary DNF of f. Then $C^*_\vee(T(f))$ can be computed from φ in $O(n^2m^2)$ time, where m is the number of terms in φ.*

Proof. Suppose without loss of generality that no term in φ is empty. By Lemma 6.2, $C^*_\vee(T(f))$ can be obtained by computing all $\vee$-extreme vectors of $T_-(\varphi)$. Clearly, $|T_-(\varphi)| \leq n|\varphi|$, and $T_-(\varphi)$ can be constructed from φ in $O(n^2m)$ time. Clearly, for every $v \in T_-(\varphi)$ it holds that v is an $\vee$-extreme vector of $T_-(\varphi)$ if and only if

$$v \neq \bigvee \{w \in T_-(\varphi) \mid w \leq v\}. \tag{6.17}$$

Let for every term t be $w^{(t)}$ the vector such that $ON(w^{(t)}) = ON(v) \setminus N(t)$. We claim that (6.17) is equivalent to

$$v \neq \bigvee \{w^{(t)} \mid t \in \varphi, P(t) \subseteq ON(v)\}. \tag{6.18}$$

Indeed, since $\{w \in T_-(\varphi) \mid w \leq v\} = \cup_{t \in \varphi \,:\, P(t) \subseteq ON(v)} \{w \in T_-(t) \mid w \leq v\}$, we have

$$\begin{aligned} \bigvee \{w \in T_-(\varphi) \mid w \leq v\} &= \bigvee_{t \in \varphi \,:\, P(t) \subseteq ON(v)} \left(\bigvee \{w \in T_-(t) \mid w \leq v\}\right) \\ &= \bigvee \{w^{(t)} \mid t \in \varphi, P(t) \subseteq ON(v)\}, \end{aligned}$$

which proves our claim.

We can easily see that computing all $w^{(t)}$ such that $t \in \varphi$ and $P(t) \subseteq ON(v)$ can be done in $O(nm)$ time, and the $\vee$-extremeness of $v \in T_-(\varphi)$ can be checked by (6.18) in $O(nm)$ time. Since $C^*_\vee(T_-(\varphi)) \subseteq T_-(\varphi)$ and $|T_-(\varphi)| \leq nm$, it follows that $C^*_\vee(T_-(\varphi)) = C^*_\vee(T(f))$ can be obtained in $O(n^2m^2)$ time. □

Corollary 6.4 *Let φ be an arbitrary DNF for a bidual Horn function f of n variables. Then $C^*_\vee(T(f))$ can be computed from φ in $O(n^2m^2)$, where m is the number of terms in φ.* □

Let us next consider the converse process of computing a DNF formula for a bidual Horn function from its characteristic set. For any function f and vector v, let $U_f(v) = \{w \in T(f) \mid v \leq w, |ON(w)| \leq |ON(v)| + 1\}$, i.e., $U_f(v)$ contains v and all true vectors of f resulting from v by switching one 0 to 1.

Lemma 6.5 *Let f be a bidual Horn function with characteristic set $S = C^*_\vee(T(f))$. Then, for each $v \in S$, the term t_v such that $T_-(t_v) = U_f(v)$ is Horn and satisfies $t_v \leq f$.*

Proof. Consider for any $v \in S$ the term t_v. Lemma 6.1 implies that for every vector $u \in T(t_v)$, there is some $S_u \subseteq T_-(t_v)$ such that $u = \bigvee S_u$. Since $T_-(t_v) = U_f(v) \subseteq T(f)$, we have $u \in Cl_\vee(T_-(t_v)) = Cl_\vee(T(f)) = T(f)$; it follows $t_v \leq f$. Furthermore, the term t_v is Horn. To see this, suppose to the contrary it is not Horn. This implies that there exist components i and j, $i \neq j$, such that $w_i = w_j = 0$ for every $w \in U_f(v)$. Let for any k be $v^{(k)}$ the vector such that $ON(v^{(k)}) = ON(v) \cup \{k\}$. Then, clearly $v^{(i)}, v^{(j)} \notin U_f(v)$, and hence $v^{(i)}, v^{(j)} \in F(f)$. Since f is Horn, it follows $v^{(i)} \wedge v^{(j)} = v \in F(f)$. However, this is a contradiction, which shows t_v is Horn. □

Lemma 6.6 *Let f be a bidual Horn function with characteristic set $S = C^*_\vee(T(f))$. Then the following Horn DNF φ represents f:*

$$\varphi \;=\; \bigvee_{v \in S} t_v \vee \bigvee_{v,w \in S\,:\,|N(t_v) \cup N(t_w)|=2} t^+_{v,w}, \tag{6.19}$$

where t_v is the term such that $T_-(t_v) = U_f(v)$, and $t^+_{v,w}$ is the positive term such that $P(t^+_{v,w}) = P(t_v) \cup P(t_w)$ (cf. Section 3).

Proof. Lemma 6.5 tells that φ of (6.19) is Horn. Hence, by Lemma 3.4, φ represents a bidual Horn function. Moreover, Lemma 3.3 and Lemma 6.5 imply $T(\varphi) \subseteq T(f)$. To prove the result, it thus remains to show $T(f) \subseteq T(\varphi)$. To prove this, assume to the contrary that there exists a vector $u \in T(f) \setminus T(\varphi)$. We will derive a contradiction.

Since $u \in T(f)$, there exists some $S_u \subseteq S$ such that $u = \bigvee S_u$. Fix any such S_u, and let for every $w \in S_u$ be t_w the term such that $T_-(t_w) = U_f(w)$. Note that, by Lemma 6.5, each t_w is a Horn term. Then we have the following three cases.

(a) Some t_w is positive. Then, $t_w(u) = 1$ holds as $u \geq w$. Since t_w occurs in φ, this implies $u \in T(\varphi)$. This is a contradiction.

(b) The terms t_w have a common negative literal $\overline{x}_j$. Then, every t_w satisfies $t_w(u) = 1$ because $u_j = 0$ and $u \geq w$. As in case (a) it follows $u \in T(\varphi)$, which is a contradiction.

(c) There are terms t_w, t_v such that $|N(t_v) \cup N(t_w)| = 2$. Since $v, w \leq u$,

$$P(t_v) \;=\; ON(v) \subseteq ON(u) \quad \text{and} \quad P(t_w) \;=\; ON(w) \;\subseteq\; ON(u);$$

hence, $P(t^+_{v,w}) = P(t_v) \cup P(t_w) \subseteq ON(u)$, which implies $t^+_{v,w}(u) = 1$. Since $t^+_{v,w}$ occurs in φ, it follows $u \in T(\varphi)$, which is a contradiction.

By (a), (b) and (c), the existence of $u \in T(f) \setminus T(\varphi)$ always leads to a contradiction. This proves $T(f) \subseteq T(\varphi)$ and hence the result. $\square$

Exploiting this lemma, we obtain the following result.

Theorem 6.7 *Let f be a bidual Horn function of n variables. Then a Horn DNF of f can be constructed from $C^*_\vee(T(f))$ in $O(n^2|C^*_\vee(T(f))|^2)$ time.*

Proof. Let $S = C^*_\vee(T(f))$. Then, for every $v \in S$, the set $U_f(v)$ can be computed in $O(n^2|S|)$ time, since checking if $w \in Cl_\vee(S)$ for each w such that $v \leq w$ and $|ON(w)| = |ON(v)| + 1$ can be done in $O(n|S|)$ time, and there are at most n such w's. Thus the collection of all t_v's in (6.19) can be computed in $O(n^2|S|^2)$ time. Furthermore, the collection of all $t^+_{v,w}$'s can be computed in $O(n|S|^2)$ time because the number of pairs of v and w is at most $|S|^2$. In total, the computation of φ in (6.19) takes $O(n^2|S|^2)$ time, which proves the result. $\square$

The following theorem shows a relation between the sizes of the characteristic set and the number of terms in term-shortest Horn DNFs for bidual Horn functions.

Theorem 6.8 *Let f be a bidual Horn function of n variables different from tautology, and let m^* be the number of terms in term-shortest DNF for f. Then*

$$\frac{|C^*_\vee(T(f))|}{n} \;\leq\; m^* \;\leq\; |C^*_\vee(T(f))| + |C^*_\vee(T(f))|^2. \tag{6.20}$$

Proof. Apply Lemmas 6.2 and 6.6. $\square$

The lower bound for m^* in Theorem 6.8 holds for general functions f whose set of true vectors $T(f)$ is closed under union. However, the upper bound does not hold for such functions in general, because m^* may be exponential with respect to $|C^*_\vee(T(f))|$.

Let us finally consider the dualization of a bidual Horn function f, which is the task of computing an irredundant prime Horn DNF ψ for f^d from a given Horn DNF φ for f. We note at this point that $|\psi|$ may be exponential with respect to $|\varphi|$. For example, let f be a positive function represented by $\varphi = \bigvee_{i=1}^n x_i x_{n+i}$ of $2n$ variables, which is clearly bidual Horn. In this case, f^d is represented by $\psi = \bigvee_{i_j \in \{j,n+j\}} x_{i_1} \cdots x_{i_n}$. We can see that φ and ψ are the unique prime DNFs for f and f^d, respectively, and that $|\varphi|\ (= \Theta(n)) \ll |\psi|\ (= \Theta(2^n \cdot n))$. In such cases, the running time of a dualization algorithm A is usually measured by its input size and output size, and A is called *polynomial total time* if its running time is polynomial in the combined input size $|\varphi|$ and output size $|\psi|$ [23, 4, 11].

An interesting problem with this respect is dualization of a positive function, where the input formula is a positive DNF φ. Many practical problems are known to be equivalent to this problem [11], but it is not known to date whether it has a polynomial total time algorithm or not [4, 11, 23, 15]. However, the recent result by Fredman and Khachiyan [15] shows that the problem is solvable in $O(m^{o(\log m)})$ time, where m is the number of terms in ψ and φ.

From results in [24, 26], which have been found in similar form in the database domain [11], the following is known about general Horn functions.

Proposition 6.9 *Let f be a Horn function.*

(1) *There is a polynomial total time algorithm for computing all prime implicants of f from $C^*_\vee(T(f^d))$ if and only if there is a polynomial total time algorithm for dualizing a positive function.*

(2) *Computing an irredundant Horn DNF of f from $C^*_\vee(T(f^d))$ is at least as hard as dualizing a positive function.*

(3) *Computing $C^*_\vee(T(f^d))$ from a Horn DNF of f is at least as hard as dualizing a positive function.* □

It is believed that the problems in (2) and (3) are strictly harder than dualizing a positive function (cf. [24]). For bidual Horn functions f, however, we can show that they have the same complexity.

Theorem 6.10 *Let f be a bidual Horn function. Then there is a polynomial total time algorithm for computing an irredundant prime Horn DNF of f^d from $C^*_\vee(T(f))$ if and only if there is a polynomial total time algorithm for dualizing a positive function.*

Proof. Note that f^d is bidual Horn if f is bidual Horn. Since every positive function is bidual Horn, Theorem 6.3 implies the only-if direction. To prove the if-direction, it is by part (1) of Proposition 6.9 sufficient to show that a polynomial total time algorithm for computing all prime implicants of f^d from $C^*_\vee(T(f))$ implies a polynomial total time algorithm for computing an irredundant prime Horn DNF of f^d from $C^*_\vee(T(f))$. Indeed, this holds for bidual Horn functions f: Since f^d is bidual Horn, by Theorem 5.13, the number of all prime implicants of f^d is polynomial in the size of a literal-shortest DNF for f^d, and computing an irredundant prime DNF from any Horn DNF is polynomial [19]. □

Corollary 6.11 *For bidual Horn functions f, each of the following problems has a polynomial total time algorithm if and only if there is a polynomial total time algorithm for dualizing a positive function:*

(i) *Computing $C^*_\vee(T(f^d))$ from $C^*_\vee(T(f))$.*

(ii) *Computing $C^*_\vee(T(f^d))$ from a Horn DNF of f.*

(iii) *Computing an irredundant prime Horn DNF of f^d from a Horn DNF of f (i.e., dualization of a bidual Horn function).* □

The above results tell that transformation problems seem to become easier if functions are restricted to be bidual Horn.

7 Conclusion and Further Research

In this paper, we have introduced bidual Horn functions, which are Boolean functions f such that both f and its dual f^d are Horn. This class of functions is motivated by the unbalanced treatment of positive and negative information through Horn functions in the extension problem of partially defined Boolean functions (pdBfs). We also emphasize that bidual Horn functions are natural generalization of positive functions with respect to the closure properties.

We have studied the semantical and computational aspects of bidual Horn functions, focusing on the recognition problem, i.e., deciding whether a given (possibly restricted) formula φ represents such a function, and on the extension problem, i.e., deciding whether for a given pdBf (T, F) a bidual Horn function exists that interpolates on (T, F). In the course of this investigation, we have determined characterizations and properties of bidual functions.

The class of bidual Horn functions appears to be an interesting intermediate class between the classes of positive and Horn functions. As for positive functions, any irredundant prime DNF is a term-shortest DNF for a bidual Horn function, but it is no longer unique. Besides a term-shortest DNF, also a literal-shortest DNF for a bidual Horn function can be computed from a Horn DNF in polynomial time; both problems are NP-hard for arbitrary Horn functions. Thus, bidual Horn functions are a nontrivial restriction of Horn functions for which these problems are polynomial. Furthermore, we have shown that dualizing a bidual Horn function is polynomially equivalent to dualizing a positive function. For the extension problem, we have presented an algorithm which decides about existence of a bidual Horn extension for a pdBf (T, F) in $O(n|T||F|)$ time and outputs a DNF in $O(n|T|(|T|+|F|))$ time. Moreover, we have shown that finding a term-shortest or literal-shortest bidual Horn extension is NP-hard, and that a polynomial algorithm for deciding whether (T, F) has a unique bidual Horn extension is difficult to find.

Our results show that from the computational point of view, bidual Horn functions benign generalize positive functions to a subclass of Horn functions. Further issues on bidual Horn functions are considered in the extended report [12]. In particular, the closure of $\mathcal{C}_{BH}$ under renamings (i.e., a change in polarity of part of the variables) is considered there. While the recognition problem is still polynomial, the extension problem is intractable.

Possible topics of future research are the development of a good algorithm for enumerating all bidual Horn extensions, as well as approximation of a term-shortest or literal-shortest bidual Horn extension.

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References

[1] D. Angluin, Queries and concept learning, *Machine Learning*, 2 (1988) 319-342.

[2] D. Angluin, M. Frazier, and L. Pitt, *Learning conjunctions of Horn clauses*, 31st Annual Symposium on Foundations of Computer Science (1990) 186-192.

[3] G. Ausiello, A. D'Atri and D. Sacca, Minimal representation of directed hypergraphs, *SIAM J. on Computing*, 15 (1986) 418-431.

[4] J. C. Bioch and T. Ibaraki, Complexity of identification and dualization of positive Boolean functions, *Information and Computation*, 123 (1995) 50-63.

[5] E. Boros, V. Gurvich, P. L. Hammer, T. Ibaraki, and A. Kogan, Decomposability of partially defined Boolean functions, *Discrete Applied Mathematics*, 62 (1995) 51-75.

[6] E. Boros, T. Ibaraki and K. Makino, Error-free and best-fit extensions of partially defined Boolean functions, *Information and Computation*, 140 (1998) 254-283.

[7] S. Ceri, G. Gottlob, L. Tanca, *Logic Programming and Databases*, Springer, 1990.

[8] Y. Crama, P. L. Hammer and T. Ibaraki, Cause-effect relationships and partially defined Boolean functions, *Annals of Operations Research*, 16 (1988) 299-326.

[9] D. W. Dowling and J.H. Gallier, Linear-time algorithms for testing the satisfyability of propositional Horn formulae, *Journal of Logic Programming*, 3 (1984) 267-284.

[10] R. Dechter and J. Pearl, Structure identification in relational data, *Artificial Intelligence*, 58 (1992) 237-270.

[11] T. Eiter and G. Gottlob, Identifying the minimal transversals of a hypergraph and related problems, *SIAM J. on Computing*, 24 (1995) 1278-1304.

[12] T. Eiter, T. Ibaraki and K. Makino, On Bidual Horn Functions, RUTCOR Research Reports RRR 19-97, September 1997.

[13] T. Eiter, T. Ibaraki and K. Makino, Double Horn functions,*Information and Computation*, 142 (1998).

[14] O. Ekin, P.L. Hammer and U. N. Peled, Horn functions and submodular Boolean functions, *Theoretical Computer Science*, 175 (1997) 257-270.

[15] M. Fredman and L. Khachiyan, On the complexity of dualization of monotone disjunctive normal forms, *Journal of Algorithms*, 21 (1996), 618-628.

[16] M. Golumbic, P.L. Hammer, P. Hansen, and T. Ibaraki (eds), Horn Logic, search and satisfiability, *Annals of Mathematics and Artificial Intelligence*, vol. 1 (1990).

[17] M. R. Garey and D. S. Johnson, *Computers and Intractability*, Freeman, New York, 1979.

[18] P. L. Hammer and A. Kogan, Optimal compression of propositional Horn knowledge bases: Complexity and approximation, *Artificial Intelligence*, 64 (1993) 131-145.

[19] P. L. Hammer and A. Kogan, Horn functions and their DNFs, *Information Processing Letters*, 44 (1992) 23-29.

[20] P. L. Hammer and A. Kogan, Quasi-acyclic propositional Horn knowledge bases: optimal compression, *IEEE Transactions on Knowledge and Data Engineering*, 7 (1995) 751-762.

[21] W. Hodges, Logical features of Horn clauses, in: *Handbook of Logic in Artificial Intelligence and Logic Programming I: Logical Foundations*, D.M. Gabbay, C.J. Hogger, and J.A. Robinson (eds), Clarendon Press, Oxford UK, 1993.

[22] A. Itai and J. Makowsky, Unification as a complexity measure for logic programming, *Journal of Logic Programming*, 4 (1987) 105-117.

[23] D. S. Johnson, M. Yannakakis, and C. H. Papadimitriou, On generating all maximal independent sets, *Information Processing Letters*, 27 (1988) 119-123.

[24] R. Khardon, Translating between Horn representations and their characteristic models, *Journal of Artificial Intelligence Research*, 3 (1995) 349–372.

[25] H. A. Kautz, M. J. Kearns, and B. Selman, Horn approximations of empirical data, *Artificial Intelligence*, 74 (1995) 129-145.

[26] D. Kavvadias, C. H. Papadimitriou, and M. Sideri, On Horn envelopes and hypergraph transversals, *ISAAC'93* Algorithms and Computation, edited by K. W. Ng et al., Springer Lecture Notes in Computer Science, 762 (1993) 399-405.

[27] D. Maier, Minimal covers in the relational database model, *Journal of the ACM*, 27 (1980) 664-674.

[28] K. Makino, K. Hatanaka and T. Ibaraki, Horn extensions of a partially defined Boolean function, RUTCOR Research Report RRR 27-95, Rutgers University, 1995.

[29] K. Makino, K. Yano and T. Ibaraki, Positive and Horn decomposability of partially defined Boolean functions, *Discrete Applied Mathematics*, 74 (1997), 251-274.

[30] W. Quine, A way to simplify truth functions, *American Mathematical Monthly*, 62 (1955) 627-631.

[31] B. Selman and H. Kautz, Knowledge compilation using Horn approximations, *Proceedings of the Ninth National Conference on Artificial Intelligence*, (1991) 904-909.

[32] L. G. Valiant, A theory of the learnable, Comm. ACM, 27 (1984), 1134-1142.

[33] I. Wegener, *The Complexity of Boolean Functions*, Wiley & Sons Ltd and Teubner, 1987.