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REASONING WITH PARSIMONIOUS AND MODERATELY GROUNDED EXPANSIONS

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Abstract. We investigate the complexity of autoepistemic reasoning with parsimonious and moderately grounded expansions. A stable expansion of an autoepistemic set of premises is parsimonious if its objective (i.e. nonmodal) part does not contain the objective part of any other stable expansion. We prove that deciding whether a formula φ belongs to at least one parsimonious stable expansion of a finite base set A is complete for $\mathbf{\Sigma}_3^P$, while deciding containment in *all* parsimonious stable expansions is complete for $\mathbf{\Pi}_3^P$. Similar results are derived for autoepistemic reasoning with moderately grounded expansions. In particular, we show that deciding whether a formula φ belongs to some moderately grounded expansion of a finite base set A is $\mathbf{\Sigma}_3^P$-complete, and that deciding whether φ belongs to all moderately grounded expansions is $\mathbf{\Pi}_3^P$-complete. These results suggest that reasoning with parsimonious stable expansions and moderately grounded expansions is strictly harder than reasoning in Moore's standard version of autoepistemic logic. We also address the complexity of reasoning if the set A is in a normalized form, and derive completeness results for this case.

1 Introduction

In this paper we study the complexity of decision problems for variants of Moore's Autoepistemic Logic (AEL) [20]. AEL is known as a successful tool for formalizing principles of nonmonotonic reasoning. This logic is based on the language of propositional logic extended by a modal belief operator L. Informally, if φ is a formula (possibly containing occurrences of L), $L\varphi$ means *φ is believed* or also, *φ is in the knowledge base*, where the knowledge base is supposed to contain the set of all beliefs of an ideally rational introspective agent.

In AEL, each given set A of initial beliefs is mapped to a set of expansions, where each expansion is an alternative possible set of total beliefs based on A. The main inference tasks of AEL are to decide whether a given formula φ occurs in at least one stable expansion of A (*brave reasoning*), and to determine whether a given formula φ occurs in all stable expansions of A (*cautious reasoning*). The complexity of these problems for a finite set A was investigated in [9], where it was shown that

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these problems are complete for respectively $\mathbf{\Sigma}_2^P$ and $\mathbf{\Pi}_2^P$, thus harder than **NP** and co-**NP**.

In the present paper, we consider stronger versions of AEL, in particular parsimonious stable expansions and moderately grounded expansions of a finite base set A.

A natural strengthening of the concept of stable expansion is to require that the objective (i.e. nonmodal) part of a stable expansion must be minimal compared to all other stable expansions with respect to set inclusion. We refer to such stable expansions as *parsimonious*. Expansions of this type describe the minimal deductively closed objective theories that a rational agent may adopt if every accepted belief $L\varphi$ must be grounded in the agent's knowledge base, i.e. φ must be derivable from the knowledge base and the beliefs.

Our investigation into the complexity of reasoning with parsimonious stable expansions covers the main decision problems in autoepistemic reasoning. In particular, we show that deciding whether a formula φ belongs to at least one parsimonious stable expansion of a finite base set A is complete for $\mathbf{\Sigma}_3^P$, while deciding for containment in *all* parsimonious stable expansions is complete for $\mathbf{\Pi}_3^P$. However, deciding whether A has any parsimonious stable expansion is $\mathbf{\Sigma}_2^P$-complete, and hence of the same complexity as deciding whether A has any stable expansion.

Similar results are shown for reasoning with *moderately grounded expansions*, which were introduced in [11] (see [31] for another development of the same concept). Informally, a moderately grounded expansion of A is a stable expansion of A whose objective part is minimal over the objective parts of stable belief sets that include A. Moderately grounded expansions correspond to a more conservative concept of believing which, in particular contexts, is more appropriate than the one corresponding to standard AEL [11]. Notice that moderate groundedness strengthens the concept of parsimony, since moderately grounded expansions are parsimonious, but the converse does not necessarily hold. We show that brave reasoning with the moderately grounded expansions of a finite base set A is $\mathbf{\Sigma}_3^P$-complete, while cautious reasoning is $\mathbf{\Pi}_3^P$-complete. Furthermore, we show that even deciding whether there exists any moderately grounded expansion for A is $\mathbf{\Sigma}_3^P$-complete.

Thus, unless the polynomial time hierarchy collapses at some low level, brave as well as cautious reasoning with parsimonious stable expansions or with moderately grounded expansions is strictly harder than with standard AEL expansions. The intuitive explanation for this is that minimality-checking introduces an additional source of complexity. As a consequence of our results, unless the polynomial hierarchy collapses, there cannot be any polynomial transformation from parsimonious AEL or moderately grounded AEL to standard AEL.

Our analysis also devotes attention to the complexity of autoepistemic reasoning from base sets where the knowledge is represented in some special format. In particular, we consider base sets where all formulae are of the form

$$L\varphi_1 \vee \cdots \vee L\varphi_m \vee \neg L\psi_1 \vee \cdots \vee \neg L\psi_n \vee \omega$$

where all φ_i, ψ_j and ω are objective formulae and only ω must be present. Formulae of this type have been considered in the context of moderate groundedness in [11]. Notice that this format, which is more restrictive than the Moore normal form [20, 16], only allows for formulae without nestings of L operators, and hence generates merely a small fragment of the autoepistemic language.

We show that for the case that the set A is normalized, brave reasoning with the moderately grounded expansions of such finite premise sets is $\mathbf{\Sigma}_2^P$-complete, and that cautious reasoning is $\mathbf{\Pi}_2^P$-complete.

These results are interesting since the complexity of standard AEL as well as parsimonious AEL remains unaffected by normalized premise sets, which means that in this case moderately grounded expansions are as easy as standard AEL expansions, while parsimonious stable expansions are strictly harder unless the polynomial hierarchy collapses.

The rest of the paper is organized as follows. Section 2 introduces the concepts of complexity theory necessary for our analysis and gives a brief summary of AEL and previously derived related complexity results. Section 3 contains results on parsimonious stable expansions, while Section 4 contains results on moderately grounded expansions. In Section 5, we investigate into the impact of normalized premise sets on the complexity of autoepistemic reasoning. Section 6, which concludes the paper, gives a discussion of the results and reviews related complexity results for other forms of nonmonotonic reasoning.

2 Preliminaries

2.1 Computational Complexity

We start with a brief review of the relevant concepts of complexity theory. The reader is assumed to be familiar with the basic concepts of **NP**-completeness; an excellent introduction to that field is [8]. Most of the problems we consider are **NP**-hard, but are not known to be in **NP** or a similar class such as co-**NP**. All of them reside in the polynomial hierarchy (PH), however, which has been introduced in [19] as a computational analog to the Kleene arithmetic hierarchy of recursion theory [8, 10]. The classes of PH are defined by oracle Turing machines and contain, unless the hierarchy collapses, problems of increasing complexity. They provide thus a way for characterizing the complexity of some problems harder than **NP**-complete problems, especially if completeness of a problem for some class can be shown. Problems complete for a class suffer from, depending on the class, several sources of complexity each of which leads to intractability. We succeed to establish completeness results for all considered problems.

Oracle Turing machines are ordinary Turing machines equipped with an oracle tape. Roughly speaking, the oracle tape enables the machine to check in unit time whether a string belongs to the oracle set, which is a formal language attached to the machine. Concerning decision problems, one can think of an oracle set as a "subroutine for solving a certain decision problem in unit time". $\mathbf{P}^{\mathbf{C}}$ ($\mathbf{NP}^{\mathbf{C}}$) are the decision problems solvable in polynomial time by some deterministic (nondeterministic) oracle Turing machine with an oracle set for any problem in $\mathbf{C}$. The classes $\Delta_k^P, \mathbf{\Sigma}_k^P$, and $\mathbf{\Pi}_k^P$ of PH are defined as follows:

$$\Delta_0^P = \mathbf{\Sigma}_0^P = \mathbf{\Pi}_0^P = \mathbf{P}$$

and for all $k \geq 0$,

$$\Delta_{k+1}^P = \mathbf{P}^{\Sigma_k^P}, \quad \mathbf{\Sigma}_{k+1}^P = \mathbf{NP}^{\Sigma_k^P}, \quad \mathbf{\Pi}_{k+1}^P = \text{co-}\mathbf{\Sigma}_{k+1}^P.$$

In particular, $\mathbf{NP} = \mathbf{\Sigma}_1^P$, co-$\mathbf{NP} = \mathbf{\Pi}_1^P$, $\mathbf{\Sigma}_2^P = \mathbf{NP}^{\mathbf{NP}}$, and $\mathbf{\Sigma}_3^P = \mathbf{NP}^{\mathbf{NP}^{\mathbf{NP}}}$. Note that $\Delta_2^P = \mathbf{P}^{\mathbf{NP}}$ is the class of problems decidable in deterministic polynomial time with an $\mathbf{NP}$ oracle set. PH is equal to $\bigcup_{k=0}^{\infty} \mathbf{\Sigma}_k^P$. We say that a problem is at the k-th level of PH iff it is complete for Δ_{k+1}^P under Turing reductions (i.e., it is in Δ_{k+1}^P and $\mathbf{\Sigma}_k^P$-hard or $\mathbf{\Pi}_k^P$-hard).

A well-known problem at the k-th level of PH, $k \geq 1$, is deciding the validity of a quantified Boolean formula (QBF) with k "quantifier alternations". A QBF is a sentence of the form $Q_1 x_1 \cdots Q_n x_n F$, where F is a propositional formula on pairwise distinct variables $x_1, \ldots, x_n$ and $Q_i \in \{\forall, \exists\}$ is a quantifier ranging over $\{\mathit{false}, \mathit{true}\}$.[1] Such a formula has k quantifier alternations if the quantifier prefix $Q_1 \cdots Q_n$ is of type $\exists \cdots \exists \forall \cdots \forall \exists \cdots \exists \cdots$ or $\forall \cdots \forall \exists \cdots \exists \forall \cdots \forall \cdots$ with k alternating quantifier groups, i.e. k is the maximum number such that there exist $1 \leq i_1 < i_2 < \cdots < i_{k-1} < n$ with $Q_{i_j} \neq Q_{i_j+1}$, for all $1 \leq j \leq k-1$. Deciding if a QBF Φ satisfies $\Phi \in \mathrm{QBF}_{k,\exists}$ ($\Phi \in \mathrm{QBF}_{k,\forall}$), where $\mathrm{QBF}_{k,\exists}$ ($\mathrm{QBF}_{k,\forall}$) denotes the set of valid QBFs with k quantifier alternations and $Q_1 = \exists$ ($Q_1 = \forall$) is well-known to be $\mathbf{\Sigma}_k^P$-complete ($\mathbf{\Pi}_k^P$-complete). Recall that a problem Π is complete for a class $\mathbf{C}$ of decision problems iff Π belongs to $\mathbf{C}$ and is $\mathbf{C}$-hard, i.e., each problem Π' in $\mathbf{C}$ is polynomial time transformable into Π.

2.2 Autoepistemic Logic (AEL)

$\mathcal{L}$ is assumed to be an ordinary language of well-formed propositional formulae over a countable set of propositional variables, built with syntactic operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, and $\bot$, where $\top$ and $\bot$ are constants for truth and falsity, respectively. The autoepistemic language $\mathcal{L}_L$ is the expansion of $\mathcal{L}$ obtained by adding a unary modal operator L, which is an "introspective" operator referring to the knowledge of a rational agent. Intuitively, a formula $L\varphi$ means that the formula φ *is believed*, i.e., is assumed to be valid. Note that nested occurrences of L are possible; $LL\varphi$ means that the agent believes in his belief in φ.

For our complexity study, we assume that the knowledge base of the agent is given by a finite set of formulae from $\mathcal{L}_L$.

Definition 2.1 *A set of (autoepistemic) premises is a finite subset of* $\mathcal{L}_L$.

The letter A will be reserved to denote a set of premises throughout the rest of this paper. Autoepistemic logic in its general setting also respects infinite knowledge bases, which is beyond the scope of our analysis.

Within $\mathcal{L}_L$, formulae from $\mathcal{L}$ are called *objective* formulae. For each set $S \subseteq \mathcal{L}_L$, we denote by $P(S)$ the *objective part* $S \cap \mathcal{L}$ of S, i.e., the objective formulae in S.

Interpretations of $\mathcal{L}_L$ are defined as ordinary propositional interpretations where formulae of the form $L\varphi$ are considered as atoms. More precisely, the atoms of a formula are all propositional atoms plus all "modal" atoms, which are all subformulae $L\psi$ which do not occur in the scope of an L operator. For example, $p \vee LL(p \wedge q)$ has the atoms p and $LL(p \wedge q)$. An interpretation assigns each formula from $\mathcal{L}_L$ a truth value by the classical rules of truth recursion, based upon truth values for the atoms; $\varphi \in \mathcal{L}_L$ ($S \subseteq \mathcal{L}_L$) is satisfied in this interpretation iff φ is true (all formulae

[1] In fact, Quantified Propositional Formula (QPF) rather than QBF would be correct. Note that QPFs are closed second-order formulae. In abuse of terminology, we do not distinguish between the isomorphic concepts of QPF and QBF.

in S are true). The consequence relation $\models$ is defined as follows. If $S \subseteq \mathcal{L}_L$ and $\varphi \in \mathcal{L}_L$, then $S \models \varphi$ iff φ is satisfied in all interpretations which satisfy S. The consequence operator *cons* is defined as $cons(S) = \{\varphi \in \mathcal{L}_L : S \models \varphi\}$. Note that classical propositional logic is naturally extended from $\mathcal{L}$ to $\mathcal{L}_L$.

The main objective of autoepistemic logic (AEL) is, as mentioned above, to model introspective knowledge. The alternative belief sets that an ideal agent may adopt from a set of premises are called the *stable expansions* of the premises; they can formally be defined by a fixed point equation as follows.

Definition 2.2 [20] *$E \subseteq \mathcal{L}_L$ is a stable expansion of a set of premises A iff*

$$E = cons(A \cup \{L\varphi : \varphi \in E\} \cup \{\neg L\varphi : \varphi \notin E\}).$$

Thus, a belief set of an ideal agent contains the premises A and the belief in everything which is in the belief set ($L\varphi$ for $\varphi \in E$) and no belief in anything which is not in the belief set ($\neg L\varphi$ for $\varphi \notin E$). The concept of a stable expansion is stronger than the concept of a stable set, which is a type of a belief set defined as follows.

Definition 2.3 *A set $S \subseteq \mathcal{L}_L$ is called stable if S satisfies the following three conditions:*

(i) $S = cons(S)$,

(ii) $\varphi \in S \Rightarrow L\varphi \in S$,

(iii) $\varphi \notin S \Rightarrow \neg L\varphi \in S$.

Every stable expansion of a premise set A is a stable set containing A, but the converse does not hold in general. For example, consider the set $A = \{Lp\}$ where p is a propositional letter. A has no stable expansion since the belief Lp can not be grounded in the premises. $\mathcal{L}_L$ is a stable set, however, which contains Lp.

It is well-known that every stable set S is uniquely characterized by its objective part $P(S)$, and that for each set of objective formulae $T \subseteq \mathcal{L}$ there exists a unique stable set $S \supseteq T$ such that $P(S) = P(cons(T))$ [20, 12]. This stable set is denoted by $E(T)$. If a premise set A is objective, then $E(A)$ is the only stable expansion of A.

Definition 2.4 *The partial order $\preceq$ on the stable sets of $\mathcal{L}_L$ is defined by $S_1 \preceq S_2$ iff $P(S_1) \subseteq P(S_2)$.*

As usual, we write $S_1 \prec S_2$ for $S_1 \preceq S_2 \wedge S_1 \neq S_2$. For example, consider $A = \{Lp \rightarrow p\}$. A has two stable expansions: $E(\{p\})$ and $E(\emptyset)$. Notice that $E(\emptyset) \prec E(\{p\})$.

Several finitary characterizations of the stable expansions of a set of premises appear in the literature [29, 16, 24]. We use here the criterion by Niemelä in [24]. For any formula $\varphi \in \mathcal{L}_L$ denote by φ^L the set of all subformulae $L\psi$ of φ. Furthermore, for $S \subseteq \mathcal{L}_L$, let $S^L = \bigcup_{\varphi \in S} \varphi^L$, and let $S^{\pm L} = \{L\psi, \neg L\psi : L\psi \in S^L\}$.

Definition 2.5 *For a set A of premises, $\Lambda \subseteq A^{\pm L}$ is A-full iff both conditions (i) and (ii) hold for each $L\varphi \in A^L$: (i) $A \cup \Lambda \models \varphi$ iff $L\varphi \in \Lambda$ (ii) $A \cup \Lambda \not\models \varphi$ iff $\neg L\varphi \in \Lambda$.*

Note that if Λ is A-full, then for each $L\varphi \in A^L$, either $L\varphi \in \Lambda$ or $\neg L\varphi \in \Lambda$.

Proposition 2.1 [24] *Given a premise set A and $K \subseteq A^{\pm L}$, deciding whether K is A-full is in Δ_2^P.*

The one-to-one correspondence between stable expansions and A-full sets is well-known.

Proposition 2.2 [24] *For each stable expansion E of a set of premises A, there exists a uniquely determined A-full set Λ, given by $\Lambda = A^{\pm L} \cap (\{L\varphi : L\varphi \in E\} \cup \{\neg L\varphi : \varphi \notin E\})$. Conversely, each A-full set Λ induces a unique stable expansion of A.*

The stable expansion corresponding to the A-full set Λ is denoted by $SE_A(\Lambda)$, and the A-full set corresponding to the stable expansion E, which is called the *kernel* of E, by $\Lambda_A(E)$. It is immediate from Proposition 2.2 that $\Lambda_A(\mathcal{L}_L) = A^L$ for every premise set A.

Logical consequence of a formula from a stable expansion of a set of premises can be described as follows. (Note: since stable expansions are deductively closed, logical consequence is equivalent to membership in the expansion).

Let $QS(\varphi)$ denote the set of all *quasi-subformulae* of $\varphi \in \mathcal{L}_L$. Quasi-subformulae are defined as subformulae in the usual way except that every formula $L\psi$ does not have genuine subformulae. For example, $\varphi = L(p \vee Lq) \wedge (Lp \vee LLr)$ has $QS(\varphi) = \{\varphi, L(p \vee Lq), Lp \vee LLr, Lp, LLr\}$.

Proposition 2.3 [24] *Let E be a stable expansion of a set A of premises and $\varphi \in \mathcal{L}_L$. Let $B = \{L\psi \in QS(\varphi) : \psi \in E\}$, $C = \{\neg L\psi : L\psi \in QS(\varphi), \psi \notin E\}$. Then, $\varphi \in E$ iff $A \cup \Lambda_A(E) \cup B \cup C \models \varphi$.*

In the case $\varphi \in \mathcal{L}$ we have $B = C = \emptyset$, and hence $\varphi \in E$ iff $A \cup \Lambda_A(E) \models \varphi$. In particular, $\bot \in E$, i.e. $E = \mathcal{L}_L$, iff $A \cup \Lambda_A(E)$ is not consistent. Since $\Lambda_A(\mathcal{L}_L) = A^L$ and A^L is A-full if $A \cup A^L$ is not consistent, it holds that $\mathcal{L}_L$ is a stable expansion of A iff $A \cup A^L$ is not consistent.

Proposition 2.3 leads to the following upper bound for deciding whether a formula belongs to a particular stable expansion E of a premise set A.

Corollary 2.4 [24] *Given a set of premises A and the kernel Λ of a stable expansion E of A, deciding if $\varphi \in \mathcal{L}_L$ belongs to E is in Δ_2^P.*

The three main decision problems in autoepistemic reasoning are

- (i) deciding whether A has a stable expansion,
- (ii) deciding whether a formula φ belongs to some stable expansion of A (*brave reasoning*),
- (iii) deciding whether a formula φ belongs to all stable expansions of A (*cautious reasoning*).

Recently, Gottlob presented a precise complexity characterization of those problems.

Proposition 2.5 [9] *Given a set of premises A, (i) deciding whether A has a stable expansion is $\mathbf{\Sigma}_2^P$-complete; (ii) deciding whether a formula φ belongs to some stable expansion of A is $\mathbf{\Sigma}_2^P$-complete; and (iii) deciding whether a formula φ belongs to all stable expansions of A is $\mathbf{\Pi}_2^P$-complete.*

Not all computational issues are answered by these results, however. Several researchers have argued that stable expansions are too permissive for modeling the belief sets of an ideal agent, since the derivation of facts from "ungrounded" beliefs may occur [11, 21, 14, 15, 23]. For example, the expansion $E(\{p\})$ of $\{Lp \to p\}$ seems defeasible, since p can only be derived on behalf of the belief Lp in p. Stronger concepts of groundedness for expansions have been proposed in [11, 21, 15, 23].

The notion of strongly grounded expansions [11] and the equivalent concepts of strongly iterative and robust expansions [14] are syntax dependent and only defined for premise sets in normal form.

A promising approach are iterative expansions of Marek and Truszczyński [14], which strengthen the concept of stable expansions. (See also [23] for an elegant enumeration-based characterization.) It appears that the complexity of the three main decision problems is not affected.

Proposition 2.6 [9] *Given a set of premises A, (i) deciding whether A has an iterative expansion is $\mathbf{\Sigma}_2^P$-complete; (ii) deciding whether a formula φ belongs to some iterative expansion of A is $\mathbf{\Sigma}_2^P$-complete; and (iii) deciding whether a formula φ belongs to all iterative expansions of A is $\mathbf{\Pi}_2^P$-complete.*

In the rest of this section, we derive a simple, but useful criterion for membership of an objective formula in a stable expansion, based on the kernel characterization. We first introduce additional notation.

Definition 2.6 *Let $\varphi \in \mathcal{L}_L$, and let ν be a truth assignment to (not necessarily all) atoms $a_1, \ldots, a_n$ of φ. Then, φ_ν denotes the formula that results if each occurrence of a_i in φ is replaced by $\top$ if $\nu(a_i) = true$ and by $\bot$ if $\nu(a_i) = false$, for all $1 \leq i \leq n$.*

For example, if $\varphi = p \wedge (Lp \to LLq \vee Lq)$ and $\nu(p) = true$, $\nu(Lq) = false$, then $\varphi_\nu = \top \wedge (Lp \to LLq \vee \bot)$.

Deciding whether an objective formula $\varphi \in \mathcal{L}$ belongs to E can be reduced to an implication problem on objective formulae as follows. For every stable expansion E of a set A of premises, let ν_E be the truth assignment to A^L defined by $\nu_E(L\psi) = true$ if $L\psi \in \Lambda_A(E)$ and $\nu_E(L\psi) = false$ if $L\psi \notin \Lambda_A(E)$. Then, let $F_A(E) = \{\varphi_{\nu_E} : \varphi \in A\}$. Note that $F_A(E)$ contains only objective formulae.

We can thus formulate the following easy lemma from Proposition 2.3.

Lemma 2.7 *Let E be a stable expansion of a set A of premises, and let $\varphi \in \mathcal{L}$. Then, $\varphi \in E$ iff $F_A(E) \models \varphi$.*

Proof. Since $\varphi \in \mathcal{L}$, by Proposition 2.3 $\varphi \in E$ holds iff $A \cup \Lambda_A(E) \models \varphi$. Since $(A \cup \Lambda_A(E))^L = A^L$, and for each $L\varphi \in A^L$, either $L\varphi \in \Lambda_A(E)$ or $\neg L\varphi \in \Lambda_A(E)$, it is clear that $A \cup \Lambda_A(E) \models \varphi$ iff $F_A(E) \cup \Lambda_A(E) \models \varphi$. Since no atom in $\Lambda_A(E)$ occurs in $F_A(E)$ or in φ, we have by trivial interpolation properties that $F_A(E) \cup \Lambda_A(E) \models \varphi$ iff $F_A(E) \models \varphi$. $\square$

Thus a stable expansion E of A is inconsistent, i.e. $E = \mathcal{L}_L$, if and only if $F_A(E)$ is inconsistent.

The previous lemma allows to reduce deciding whether $E' \preceq E$ for stable expansions E, E' of premise sets A, A' to a propositional implication test.

Theorem 2.8 *Let E, E' be stable expansions of respective premise sets A and A'. Then, $E' \preceq E$ if and only if $F_A(E) \models F_{A'}(E')$.*

Proof. From Lemma 2.7, we get $E' \preceq E$ iff $P(cons(F_{A'}(E'))) \subseteq P(cons(F_A(E)))$, which is equivalent to $F_A(E) \models F_{A'}(E')$. □

3 Parsimonious stable expansions

In this section, we derive complexity results for reasoning with parsimonious stable expansions, which are a natural concept of restricted stable expansions. According to Occam's *entia non sunt multiplicanda praeter necessitatem*, we restrict the stable expansions of a premise set to those whose objective part does not contain the objective part of any other stable expansion as a subset.

Definition 3.1 *A stable expansion E of a set A is parsimonious iff there exists no stable expansion E' of A such that $E' \prec E$.*

Note that in the example $A = \{Lp \to p\}$, only the stable expansion $E(\emptyset)$ is parsimonious, which eliminates the undesired stable expansion $E(\{p\})$.

The complexity of reasoning with parsimonious stable expansions has, to our best knowledge, not been considered yet. We give in this section a precise characterization of these problems in terms of completeness results for classes at the second and the third level of PH.

We start with characterizing the complexity of deciding whether $E' \prec E$ for stable expansions E, E' of respective premise sets A and A'. By Theorem 2.8, this problem can efficiently be reduced to a propositional implication test, which is a well-known intractable problem. A computationally more advantageous, that is polynomial time checkable, criterion for checking whether $E' \preceq E$ is unlikely to exist, however, since this would entail $\mathbf{NP} = \mathbf{P}$.

Theorem 3.1 *Given sets A, A' of premises and the kernels Λ, Λ' of stable expansions E, E' of A and A', respectively, deciding whether $E' \preceq E$ is co-$\mathbf{NP}$-complete.*

Proof. By Theorem 2.8, this problem is polynomially transformable into deciding whether $\varphi \models \psi$ for $\varphi, \psi \in \mathcal{L}$, which is in co-$\mathbf{NP}$. co-$\mathbf{NP}$-hardness is shown by a polynomial transformation from deciding whether $\varphi \in \mathcal{L}$ is a tautology. Without loss of generality we may assume that φ is satisfiable. Let p be a propositional variable not occurring in φ, and define $A = A' = \{Lp \to p, \neg Lp \to \varphi\}$. Then, $\Lambda = \{Lp\}$ and $\Lambda' = \{\neg Lp\}$ are A-full sets. Let $E = SE_A(\Lambda)$, $E' = SE_A(\Lambda')$ be the corresponding expansions. By Theorem 2.8, $E' \preceq E$ holds iff $F_A(E) \models F_A(E')$; since $F_A(E) \equiv p$, $F_A(E') \equiv \varphi$, this holds iff φ is a tautology. □

Note that deciding whether a premise set has a parsimonious stable expansion is not harder than deciding whether a stable expansion exists.

Proposition 3.2 *Deciding if a premise set A has a parsimonious stable expansion is $\mathbf{\Sigma}_2^P$-complete.*

Proof. Since A is finite, only finitely many A-full sets and hence only finitely many stable expansions of A exist. Hence it is clear that A has a parsimonious stable expansion iff A has any stable expansion. Since the latter problem is $\mathbf{\Sigma}_2^P$-complete [9, Theorem 4.1], the result follows. □

We consider now the problem of recognizing parsimonious stable expansions.

Theorem 3.3 *Given a set of premises A and the kernel Λ of a stable expansion E of A, deciding whether E is parsimonious is $\mathbf{\Pi}_2^P$-complete. $\mathbf{\Pi}_2^P$-hardness holds even if E is consistent.*

Proof. Membership of this problem in $\mathbf{\Pi}_2^P$ is shown as follows. E is not parsimonious if and only if there exists a stable expansion E' of A such that $E' \prec E$. A guess $\Lambda' \subseteq A^{\pm L}$ for $\Lambda_A(E')$ can be verified in polynomial time with an **NP** oracle, because checking whether Λ' is the kernel of some stable expansion E' is in Δ_2^P (Proposition 2.1) as well as deciding whether $E' \prec E$, given the respective kernels (follows from Theorem 3.1).

Hardness of this problem for $\mathbf{\Pi}_2^P$ is shown by a reduction from validity checking of a QBF $\Phi = \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l F$. We define a set of premises A as follows. Let $y'_1, \ldots, y'_m$ and s be additional variables. Let

$$\begin{aligned} A \;=\; & \{Ly_i \rightarrow y_i, L\neg y'_i \rightarrow \neg y'_i : 1 \leq i \leq m\} \cup \\ & \{y_i \wedge \neg y'_i \rightarrow s : 1 \leq i \leq m\} \cup \{s \rightarrow (y_1 \wedge \neg y'_1 \wedge \cdots \wedge y_m \wedge \neg y'_m)\} \cup \\ & \{L(\bigwedge_{i=1}^{m} (y_i \leftrightarrow y'_i) \rightarrow \neg F)\} \end{aligned}$$

Note that A is consistent and has no inconsistent expansion since $A \cup A^L$ is consistent. Let $\Lambda_0 = A^L$. It is easily verified by Definition 2.5 that Λ_0 is A-full; the corresponding stable expansion $E_0 = SE_A(\Lambda_0)$ is $E_0 = E(\{y_1, \neg y'_1, \ldots, y_m, \neg y'_m, s\})$.

For each stable expansion E of A it holds that $s \in E$ iff $E = E_0$. $s \in E$ means $\{y_i, \neg y'_i : 1 \leq i \leq m\} \subseteq E$, and since $\varphi \in E$ entails $L\varphi \in E$, $\Lambda_0 \subseteq E$ holds. Since E is consistent, $\Lambda_A(E) = \Lambda_0$ and hence $E = E_0$ follows.

We further note that each stable expansion E of A must satisfy $E \preceq E_0$. Indeed, it is not hard to see that $F_A(E_0)$ is logically equivalent to $s \wedge y_1 \wedge \neg y'_1 \wedge \cdots \wedge y_m \wedge \neg y'_m$, and since $F_A(E)$ is always satisfied if *true* is assigned to $y_1, \ldots, y_m, s$ and *false* to $y'_1, \ldots, y'_m$, we have $F_A(E_0) \models F_A(E)$. Thus by Theorem 2.8, $E \preceq E_0$.

We claim that E_0 is the only stable expansion of A (and hence clearly parsimonious) if and only if Φ is valid.

Assume Φ is not valid, that is, there exists a truth assignment ν to $y_1, \ldots, y_m$ such that $\exists z_1 \cdots \exists z_l F_\nu$ is contradictory, i.e. $\neg F_\nu$ is a tautology. Define the set K_ν as follows:

$$\begin{aligned} K_\nu \;=\; & \{Ly_i, \neg L\neg y'_i : \nu(y_i) = \mathit{true}, 1 \leq i \leq m\} \cup \\ & \{\neg Ly_i, L\neg y'_i : \nu(y_i) = \mathit{false}, 1 \leq i \leq m\} \cup \\ & \{L(\bigwedge_{i=1}^{m} (y_i \leftrightarrow y'_i) \rightarrow \neg F)\} \end{aligned}$$

Then, K_ν is A-full. To show this, we observe that $A \cup K_\nu$ is consistent with the set

$$S = \{y_i, y'_i : \nu(y_i) = \mathit{true},\ 1 \leq i \leq m\} \cup \{\neg y_i, \neg y'_i : \nu(y_i) = \mathit{false},\ 1 \leq i \leq m\} \cup \{\neg s\}.$$

Since for each Ly_i we have $Ly_i \to y_i$, we thus clearly obtain $A \cup K_\nu \models y_i$ iff $Ly_i \in K_\nu$ and $A \cup K_\nu \not\models y_i$ iff $\neg Ly_i \in K_\nu$, for $1 \leq i \leq m$. The argumentation for $L\neg y'_i$ is analogous.

It remains to verify that $A \cup K_\nu \models \bigwedge_{i=1}^m (y_i \leftrightarrow y'_i) \to \neg F$ holds. Clearly, this holds iff $A \cup K_\nu \cup \{\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i)\} \models \neg F$ holds. Since $A \cup K_\nu \cup \{\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i)\}$ is consistent and logically implies $y_i \equiv \nu(y_i)$, for $1 \leq i \leq m$, this holds iff $A \cup K_\nu \cup \{\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i)\} \models \neg F_\nu$, which is fulfilled as $\neg F_\nu$ is a tautology; thus K_ν is A-full. Since $K_\nu \neq \Lambda_0$, it follows that $SE_A(K_\nu) \neq E_0$ (and hence $SE_A(K_\nu) \prec E_0$) holds.

Conversely, assume there exists a stable expansion $E \neq E_0$ (and hence $E \prec E_0$). Then, $\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i) \to \neg F \in E$ holds, because $L(\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i) \to \neg F) \in A$, $A \subseteq E$, and for all $\psi \in \mathcal{L}_L$, $L\psi \in E$ entails $\psi \in E$ (this follows immediately from the definition of a stable expansion). Since $\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i) \to \neg F \in \mathcal{L}$, by Lemma 2.7 it follows that $F_A(E) \models \bigwedge_{i=1}^m (y_i \leftrightarrow y'_i) \to \neg F$. Clearly, this holds iff $G \models \neg F$ holds, where $G = F_A(E) \cup \{\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i)\}$. G is satisfied by the truth assignment μ to $s, y_1, y'_1, \ldots, y_m, y'_m$ defined as follows:

$$\mu(s) = \mathit{false}, \quad \mu(y_i) = \mu(y'_i) = \begin{cases} \mathit{true} & \text{if } \neg L\neg y'_i \in E \\ \mathit{false} & \text{if } L\neg y'_i \in E \end{cases}, \quad \text{for } 1 \leq i \leq m.$$

Since $G \models \neg F$, we have that for the restriction ν of μ to $y_1, \ldots, y_m$, $G_\mu \models \neg F_\nu$. Thus by trivial interpolation properties, $\models \neg F_\nu$ follows, i.e. $\forall z_1 \cdots \forall z_l \neg F_\nu$ is valid. Hence, $\exists y_1 \cdots \exists y_m \forall z_1 \cdots \forall z_l \neg F$ is valid, which means that $\Phi = \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l\, F$ is not valid. Thus the claim is proved.

It is clear that A and Λ_0 can be constructed in polynomial time. Thus we have the theorem. $\Box$

The complexity of checking if a stable expansion is parsimonious has a detrimental effect on the complexity of brave reasoning with the parsimonious stable expansions of a set of premises.

Theorem 3.4 *Deciding whether a formula $\varphi \in \mathcal{L}_L$ belongs to some parsimonious stable expansion of a set A of premises is $\mathbf{\Sigma}_3^P$-complete. $\mathbf{\Sigma}_3^P$-hardness holds even if $\varphi \in \mathcal{L}$ and every parsimonious stable expansion of A is consistent.*

Proof. Membership of this problem in $\mathbf{\Sigma}_3^P$ can be shown as follows. Guess $\Lambda \subseteq A^{\pm L}$ such that $\Lambda = \Lambda_A(E)$ and $\varphi \in E$ for some parsimonious stable expansion E of A. Since checking if Λ is the kernel of some stable expansion is in Δ_2^P (Proposition 2.1), checking whether $SE_A(\Lambda)$ is parsimonious is in $\mathbf{\Pi}_2^P$ (Theorem 3.3), and checking whether $\varphi \in E$ holds is in Δ_2^P (Corollary 2.4), the guess can be verified in polynomial time with a $\mathbf{\Sigma}_2^P$ oracle. Hence the problem is in $\mathbf{\Sigma}_3^P$.

We show $\mathbf{\Sigma}_3^P$-hardness by a polynomial transformation of validity checking of a QBF $\Phi = \exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l\, F$ into this problem. Let $y'_1, \ldots, y'_m$ and s be additional variables, and define

$$\begin{aligned} A \;=\; & \{x_i \leftrightarrow Lx_i : 1 \leq i \leq n\} \cup \{Ly_i \to y_i, L\neg y'_i \to \neg y'_i : 1 \leq i \leq m\} \cup \\ & \{y_i \wedge \neg y'_i \to s : 1 \leq i \leq m\} \cup \{s \to (y_1 \wedge \neg y'_1 \wedge \cdots \wedge y_m \wedge \neg y'_m)\} \cup \\ & \cup \{L(\bigwedge_{i=1}^m (y_i \leftrightarrow y'_i) \to \neg F)\}. \end{aligned}$$

Note that A has no inconsistent stable expansion since $A \cup A^L$ is consistent, and that this set of premises is close to A in the proof of Theorem 3.3. The only difference are the additional premises $Lx_i \leftrightarrow x_i$, $1 \leq i \leq n$, and that F can be built on $x_1, \ldots, x_n$, $y_1, \ldots, y_m$, $z_1, \ldots, z_l$ instead of $y_1, \ldots, y_m$, $z_1, \ldots, z_l$.

For every truth assignment ν to $x_1, \ldots, x_n$, the set

$$\begin{aligned}\Lambda_\nu \;=\; & \{Lx_i : \nu(x_i) = \mathit{true}, 1 \leq i \leq n\} \cup \{\neg Lx_i : \nu(x_i) = \mathit{false}, 1 \leq i \leq n\} \cup \\ & \{Ly_1, L\neg y_1', \ldots, Ly_m, L\neg y_m'\} \cup \{L(\bigwedge_{i=1}^{m} (y_i \leftrightarrow y_i') \rightarrow \neg F)\}\end{aligned}$$

is A-full. Let $E_\nu = SE_A(\Lambda_\nu)$ denote the corresponding expansion. Note that $x_i \in E_\nu$ iff $\nu(x_i) = \mathit{true}$ and $\neg x_i \in E_\nu$ iff $\nu(x_i) = \mathit{false}$, for all $1 \leq i \leq n$. Thus clearly $E_\nu \not\preceq E_{\nu'}$, $E_{\nu'} \not\preceq E_\nu$ holds iff $\nu \neq \nu'$ holds.

It is not difficult to show that every stable expansion E fulfills either $x_i \in E$ or $\neg x_i \in E$, for all $1 \leq i \leq n$; let $\tau(E)$ be the truth assignment to $x_1, \ldots, x_n$ given by E. As a consequence, $E \preceq E'$ entails that $\tau(E) = \tau(E')$, for all stable expansions E, E' of A. Furthermore, $s \in E$ iff $E = E_{\tau(E)}$ must always hold as well as $E \preceq E_{\tau(E)}$; this can be shown analogous to $s \in E$ iff $E = E_0$, and $E \preceq E_0$ in the proof of Theorem 3.3.

Since $s \in E$ implies $E = E_{\tau(E)}$, it follows that s belongs to a parsimonious stable expansion of A if and only if there exists a ν such that E_ν is parsimonious. Along the line of argumentation taken in the proof of Theorem 3.3 to prove that E_0 is parsimonious iff Φ is valid, it is straightforward to show that for each truth assignment ν to $x_1, \ldots, x_n$, E_ν is parsimonious if and only if $\forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l\, F_\nu$ is valid. Consequently, s belongs to a parsimonious stable expansion of A if and only if $\Phi = \exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l\, F$ is valid.

Clearly, A and $\varphi = s$ can be constructed in polynomial time, whence our theorem. $\Box$

An analogous result for cautious reasoning with parsimonious stable expansions can be easily derived from this result.

Theorem 3.5 *Deciding whether a formula $\varphi \in \mathcal{L}_L$ is in all parsimonious stable expansions of a set A of premises is $\mathbf{\Pi}_3^P$-complete. $\mathbf{\Pi}_3^P$-hardness even holds if every parsimonious stable expansion of A is consistent.*

Proof. Membership of the complementary problem, deciding whether φ does *not* occur in some parsimonious stable expansion of A, in $\mathbf{\Sigma}_3^P$ can be shown similar to membership of brave reasoning with parsimonious stable expansions in $\mathbf{\Sigma}_3^P$. A guess $\Lambda \subseteq A^{\pm L}$ on the kernel of a parsimonious stable expansion E of A can be verified in polynomial time with a $\mathbf{\Sigma}_2^P$ oracle (see proof of Theorem 3.4 for details). Given A and Λ, deciding whether $\varphi \notin E$ is in Δ_2^P (cf. Corollary 2.4), hence possible with one call to a $\mathbf{\Sigma}_2^P$ oracle. Consequently, deciding whether φ does not occur in some parsimonious stable expansion of A is in $\mathbf{\Sigma}_3^P$. Thus it follows that deciding whether φ occurs in all parsimonious stable expansions of A is in $\mathbf{\Pi}_3^P$.

Hardness for $\mathbf{\Pi}_3^P$ is shown from Theorem 3.4. Given $\varphi \in \mathcal{L}_L$ and A, consider the three problems of deciding whether

(i) $\neg L\varphi$ occurs in all parsimonious stable expansions of A,

(ii) $\neg L\varphi$ does not occur in all parsimonious stable expansions of A,

(iii) φ occurs in some parsimonious stable expansion of A.

If every parsimonious stable expansion of A is consistent, (ii) and (iii) are equivalent problems, and hence by Theorem 3.4 (ii) is $\mathbf{\Sigma}_3^P$-hard. It is also clear that (ii) is the complementary problem for (i); hence, (i) is $\mathbf{\Pi}_3^P$-hard, even if every parsimonious stable expansion of A is consistent. Thus the theorem follows. $\square$

While focusing on parsimonious stable expansions gets cautious reasoning from the second to the third level of the polynomial hierarchy in the general case, it is interesting to note that for purely propositional formulae, parsimony does not affect computational complexity. In particular, the following holds.

Theorem 3.6 *Let $\varphi \in \mathcal{L}$ and let A be a set of premises. Deciding whether φ belongs to all parsimonious stable expansions of A is $\mathbf{\Pi}_2^P$-complete.*

Proof. Indeed, since A has only a finite number of stable expansions, it is easily verified that φ belongs to all parsimonious stable expansions of A if and only if φ belongs to all stable expansions of A, and this problem is in $\mathbf{\Pi}_2^P$ [24]. $\mathbf{\Pi}_2^P$-hardness holds since cautious reasoning with all stable expansions is already $\mathbf{\Pi}_2^P$-hard for purely propositional formulae [9, proof of Theorem 4.5]. $\square$

4 Moderately Grounded Expansions

Another promising concept for strengthening standard AEL is the suggestion by Konolige [11] to restrict the stable expansions to moderately grounded expansions, which are the stable expansions whose objective parts do not strictly contain the objective part of any stable set that includes the premises.

Definition 4.1 *A stable expansion E of a set A is moderately grounded iff there exists no stable set S such that $A \subseteq S$ and $S \prec E$.*

Every moderately grounded expansion is parsimonious, but the converse does not hold in general. For example, if $A = \{Lp \rightarrow p, p \rightarrow q, Lq\}$, then $E(\{p,q\})$ is the only, and hence clearly parsimonious, stable expansion of A. $E(\{p,q\})$ is not moderately grounded, however, since $E(\{q\})$ is a stable set containing A and $E(\{q\}) \prec E(\{p,q\})$.

Moderately grounded expansions can also be characterized by a fixed point equation and use of modal logic.

Proposition 4.1 [11, 31]
E is a moderately grounded expansion of a set of premises A iff

$$E = cons_{\mathbf{K45}}(A \cup \{\neg L\varphi : \varphi \in \mathcal{L} - E\})$$

where $cons_{\mathbf{K45}}$ is the consequence operator of the modal logic **K45**.

We remark that Niemelä's L-hierarchic expansions [23] strengthen the concept of moderated groundedness.

The following result provides a characterization of moderately grounded expansions which is useful for a recognition algorithm.

Lemma 4.2 *Let A be a set of premises and $S \supseteq A$ be a consistent stable set. Then, $\Gamma = S \cap A^{\pm L}$ is full for $A' = A \cup \{\varphi : L\varphi \in \Gamma\}$ and $SE_{A'}(\Gamma) \preceq S$.*

Proof. Note that S is consistent, and hence either $L\varphi \in \Gamma$ or $\neg L\varphi \in \Gamma$ holds, for all $L\varphi \in A^L$.

Recall that Γ is A'-full iff for all $L\varphi \in A'^L$, (i) $A' \cup \Gamma \models \varphi$ iff $L\varphi \in \Gamma$, and (ii) $A' \cup \Gamma \not\models \varphi$ iff $\neg L\varphi \in \Gamma$. Notice that $A' \cup \Gamma \subseteq S$ and that $A'^L = A^L$.
(i): If $A' \cup \Gamma \models \varphi$, then $S \models \varphi$, hence $L\varphi \in S$ and thus $L\varphi \in \Gamma$. Conversely, if $L\varphi \in \Gamma$, then $\varphi \in \{\varphi : L\varphi \in \Gamma\}$, hence clearly $A' \cup \Gamma \models \varphi$.
(ii): Assume $A' \cup \Gamma \not\models \varphi$, but $\neg L\varphi \notin \Gamma$. Consequently, $L\varphi \in \Gamma$, and thus $\varphi \in \{\varphi : L\varphi \in \Gamma\}$. It follows $A' \cup \Gamma \models \varphi$, which is a contradiction. Hence, $A' \cup \Gamma \not\models \varphi$ implies $\neg L\varphi \in \Gamma$. Conversely, assume $\neg L\varphi \in \Gamma$. This entails $\varphi \notin S$, hence $S \not\models \varphi$. Since $A' \cup \Gamma \subseteq S$, it follows $A' \cup \Gamma \not\models \varphi$, and (ii) holds.

Now we observe that $E' = SE_{A'}(\Gamma) \preceq S$ holds: $P(E') = P(cons(A' \cup \Gamma))$ by Proposition 2.3, and since $A' \cup \Gamma \subseteq S$, $P(E') \subseteq P(S)$. $\Box$

Theorem 4.3 *A stable expansion E of a premise set A is moderately grounded iff there exists no set $\Gamma \subseteq A^{\pm L}$ such that Γ is full for $A' = A \cup \{\varphi : L\varphi \in \Gamma\}$ and $SE_{A'}(\Gamma) \prec E$.*

Proof. Recall that E is moderately grounded iff there exists no stable set S such that $S \prec E$ and $A \subseteq S$.

Since every stable expansion is a stable set, the *only if* direction clearly holds.

If E is not moderately grounded, then there exists a stable set $S \supseteq A$ such that $S \prec E$. Since S is consistent, by Lemma 4.2 $\Gamma = S \cap A^{\pm L}$ is full for $A \cup \{\varphi : L\varphi \in \Gamma\}$ and $SE_{A'}(\Gamma) \preceq S$, hence $SE_{A'}(\Gamma) \prec E$. Thus the *if* direction holds, and the result follows. $\Box$

With this result, we are able to show that recognizing moderately grounded expansions is in $\mathbf{\Pi}_2^P$. We also show that the problem is hard for this class, and hence no substantially better criterion for a recognition algorithm can be expected.

Theorem 4.4 *Given a set of premises A and the kernel Λ of a stable expansion E of A, deciding whether E is moderately grounded is $\mathbf{\Pi}_2^P$-complete.*

Proof. A guess $\Gamma \subseteq A^{\pm L}$ on the kernel of a stable expansion E' of $A' = A \cup \{\varphi : L\varphi \in \Gamma\}$ such that $E' \prec E$ can be verified in polynomial time with an **NP** oracle (Proposition 2.1, Theorem 3.1). Since by Theorem 4.3 such a Γ exists iff E is not moderately grounded, the problem is clearly in $\mathbf{\Pi}_2^P$.

We show hardness for this class by a reduction from deciding whether $\Phi \in \mathrm{QBF}_{2,\forall}$ for a QBF Φ. Let $\Phi = \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l F$. We define a set of premises A as follows. Let p be an additional variable, and define

$$A = \{Ly_1 \to y_1, \ldots, Ly_m \to y_m,\ Lp \to (y_1 \wedge \cdots \wedge y_m),\ Lp \to p,\ Lp \vee \neg G\}$$

where $G = F(Ly_1, \ldots, Ly_m, z_1, \ldots, z_l)$ is the formula obtained from F if all occurrences of the atom y_i are replaced by Ly_i, for all $1 \leq i \leq m$.

We notice that $A \cup A^L$ is consistent, hence A has only consistent stable expansions.

It is easy to verify that the set $\Lambda_0 = A^L$ is A-full, and that the corresponding stable expansion E_0 of A satisfies $E_0 = E(\{p, y_1, \ldots, y_m\})$, hence $F_A(E_0) \equiv \{p, y_1, \ldots, y_m\}$.

Furthermore, it is not hard to see that E_0 is the only stable expansion E of A such that $p \in E$, since the latter implies $\Lambda_A(E) = \Lambda_0$, hence $E = E_0$.

We claim that E_0 is not moderately grounded iff Φ is not valid.

Assume that Φ is not valid. Hence, there exists a truth assignment ν to $y_1, \ldots, y_m$ such that $\exists z_1 \cdots \exists z_l F_\nu$ is contradictory, i.e. $\neg F_\nu$ is a tautology.

Define $S = E(\{y_i : \nu(y_i) = \mathit{true}, 1 \leq i \leq n\})$. Note that $\neg Lp \in S$, and that $Ly_i \in S$ if $\nu(y_i) = \mathit{true}$ and that $\neg Ly_i \in S$ if $\nu(y_i) = \mathit{false}$, for $1 \leq i \leq m$, which entails that $\neg G \in S$. It is thus easy to see that $A \subseteq S$ holds. Since $S \prec E_0$, it follows that E_0 is not moderately grounded, and the *if* direction is shown.

Now assume E_0 is not moderately grounded. By Theorem 4.3, we know that there exists $\Gamma \subseteq A^{\pm L}$ such that Γ is full for $A' = A \cup \{\varphi : L\varphi \in \Gamma\}$ and $E' = SE_{A'}(\Gamma) \prec E_0$. This entails $p \notin E'$, and hence $\neg Lp \in E'$. Let the truth assignment μ to $y_1, \ldots, y_m$ be defined by

$$\mu(y_i) = \begin{cases} \mathit{true} & \text{if } Ly_i \in \Gamma \\ \mathit{false} & \text{if } \neg Ly_i \in \Gamma \end{cases}, \quad \text{for } 1 \leq i \leq m.$$

Consequently, $\perp \vee \neg F_\mu \in F_{A'}(E')$ holds. Since by Theorem 2.8 we have $F_A(E_0) \models F_{A'}(E')$, it follows $\{p, y_1, \ldots, y_m\} \models \neg F_\mu$. Since in F_μ only z_i variables occur, by trivial interpolation properties this holds iff $\models \neg F_\mu$ holds, i.e. $\neg F_\mu$ is a tautology. Consequently, $\exists z_1 \cdots \exists z_l F_\mu$ is contradictory, which implies that Φ is not valid. Hence the *only if* direction holds, and the claim is proved.

Since A and Λ_0 are clearly constructible in polynomial time, the theorem follows. $\square$

With this result, we can characterize the complexity of the reasoning tasks for moderately grounded expansions as follows.

Theorem 4.5 *Let A be a set of premises and let $\varphi \in \mathcal{L}_L$. Deciding whether φ occurs in some moderately grounded expansions of A is $\mathbf{\Sigma}_3^P$-complete. $\mathbf{\Sigma}_3^P$-hardness holds even if $\varphi \in \mathcal{L}$ and every moderately grounded expansion of A is consistent.*

Proof. A guess $\Lambda \subseteq A^{\pm L}$ on the kernel of a moderately grounded expansion E of A can be verified in polynomial time with a $\mathbf{\Sigma}_2^P$ oracle. Indeed, deciding whether Λ is the kernel of a stable expansion of A is possible with one call to a $\mathbf{\Sigma}_2^P$ oracle (cf. Proposition 2.1), and by Theorem 4.4, deciding whether the corresponding stable expansion E is moderately grounded is possible with one call to a $\mathbf{\Sigma}_2^P$ oracle. On a successful guess, deciding whether $\varphi \in E$ holds is possible with another call to a $\mathbf{\Sigma}_2^P$ oracle (cf. Corollary 2.4). It follows from this that brave reasoning with moderately grounded expansions is in $\mathbf{\Sigma}_3^P$.

The proof of $\mathbf{\Sigma}_3^P$-hardness is an extension of the construction in the proof of Theorem 4.4, which is analogously obtained as the one in the proof of Theorem 3.4. Given a QBF $\Phi = \exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_l\, F$, we construct a premise set A as follows.

$$\begin{aligned} A \;=\; & \{x_1 \leftrightarrow Lx_1, \ldots, x_n \leftrightarrow Lx_n\} \cup \{Ly_1 \rightarrow y_1, \ldots, Ly_m \rightarrow y_m\} \cup \\ & \{Lp \rightarrow (y_1 \wedge \cdots \wedge y_m),\ Lp \rightarrow p,\ Lp \vee \neg G\} \end{aligned}$$

where $G = F(x_1, \ldots, x_n, Ly_1, \ldots, Ly_m, z_1, \ldots, z_l)$ is the formula obtained from F if all occurrences of the atom y_i are replaced by Ly_i, for all $1 \leq i \leq m$.

Notice that A has only consistent stable expansions, as $A \cup A^L$ is consistent.

For every truth assignment ν to $x_1, \ldots, x_n$, the set

$$\begin{aligned}\Lambda_\nu &= \{Lx_i : \nu(x_i) = \mathit{true}, 1 \leq i \leq n\} \cup \{\neg Lx_i : \nu(x_i) = \mathit{false}, 1 \leq i \leq n\} \cup \\ &\quad \{Lp, Ly_1, \ldots, Ly_m\}\end{aligned}$$

is A-full, and if $E_\nu = SE_A(\Lambda_\nu)$, then $x_i \in E_\nu$ iff $\nu(x_i) = \mathit{true}$ and $\neg x_i \in E_\nu$ iff $\nu(x_i) = \mathit{false}$. Therefore, $E_\nu \not\preceq E_{\nu'}$, $E_{\nu'} \not\preceq E_\nu$ iff $\nu \neq \nu'$.

It holds that if $p \in E$ for some stable expansion E of A, then $E = E_\nu$ for some ν. Hence p belongs to some moderately grounded expansion of A iff some E_ν is moderately grounded; since E_ν is moderately grounded iff $\forall y_1 \cdots \forall y_m \exists z_1 \cdots z_l F_\nu$ is valid, p belongs to some moderately grounded expansion of A iff Φ is valid. Since A, $\varphi = p$ are constructible in polynomial time, the result follows. $\Box$

Using this result, we can easily derive that even checking for the existence of moderately grounded expansions is at the third level of the polynomial hierarchy.

Theorem 4.6 *Deciding whether a premise set A has a moderately grounded expansion is $\mathbf{\Sigma}_3^P$-complete. $\mathbf{\Sigma}_3^P$-hardness holds even if every moderately grounded expansion of A is consistent.*

Proof. Membership of this problem in $\mathbf{\Sigma}_3^P$ holds since a guess $\Lambda \subseteq A^{\pm L}$ on the kernel of a moderately grounded expansion of A can be verified in polynomial time with a $\mathbf{\Sigma}_2^P$ oracle (see proof of Theorem 4.5 for details).

Hardness for $\mathbf{\Sigma}_3^P$ is shown by a slight extension of the construction in the proof of Theorem 4.5. Recall that we constructed there a premise set A such that every moderately grounded expansion of A is consistent, and a formula φ, which is the propositional atom p, such that deciding whether φ occurs in any moderately grounded expansion of A is $\mathbf{\Sigma}_3^P$-hard.

Let q be a propositional letter not occurring in A and define $A' = A \cup \{Lp \to q, Lq\}$.

It can be easily seen that $A' \cup A'^L$ is consistent, hence A' has only consistent stable expansions. Every stable expansion E of A' must contain Lq, hence q and thus also Lp, since $Lp \to q$ is the only formula in A' that allows to derive q.

Hence, we obtain that the only A'-full sets are the sets $\Lambda'_\nu = \Lambda_\nu \cup \{Lq\}$ for each truth assignment ν to the x_i variables, with corresponding stable expansions

$$\begin{aligned}E'_\nu &= E(\{x_i : \nu(x_i) = \mathit{true}, 1 \leq i \leq n\} \cup \{\neg x_i : \nu(x_i) = \mathit{false}, 1 \leq i \leq n\} \cup \\ &\quad \{y_1, \ldots, y_m, p, q\})\end{aligned}$$

of A', which correspond one-to-one to the stable expansions

$$\begin{aligned}E_\nu &= E(\{x_i : \nu(x_i) = \mathit{true}, 1 \leq i \leq n\} \cup \{\neg x_i : \nu(x_i) = \mathit{false}, 1 \leq i \leq n\} \cup \\ &\quad \{y_1, \ldots, y_m, p\})\end{aligned}$$

of A.

By use of the interpolation theorem, it follows that there exists a stable set $S \prec E_\nu$ such that $S \supseteq A$ iff there exists a stable set $S' \prec E'_\nu$ such that $S' \supseteq A'$, for all ν. Hence, E_ν is moderately grounded for A iff E'_ν is moderately grounded for A'. Since deciding whether some E_ν is moderately grounded for A is $\mathbf{\Sigma}_3^P$-hard (which is the case iff p occurs in some moderately grounded expansion of A), it follows that deciding whether A' has any moderately grounded expansion is $\mathbf{\Sigma}_3^P$-hard. Since every moderately grounded expansion of A' is consistent, this holds under the asserted restriction; the theorem follows. □

Theorem 4.7 *Let A be a set of premises and let $\varphi \in \mathcal{L}_L$. Deciding whether φ occurs in all moderately grounded expansions of A is $\mathbf{\Pi}_3^P$-complete. This holds even if $\varphi \in \mathcal{L}$ and every moderately grounded expansion of A is consistent.*

Proof. A guess $\Lambda \subseteq A^{\pm L}$ on the kernel of a moderately grounded expansion E of A can be verified in polynomial time with a $\mathbf{\Sigma}_2^P$ oracle. On a successful guess, it can be verified in polynomial time with a **NP** oracle whether $\varphi \notin E$. Hence, cautious reasoning with moderately grounded expansions is clearly in $\mathbf{\Pi}_3^P$.

For the hardness part, note that $\perp$ occurs in all moderately grounded expansions of A iff A has no consistent moderately grounded expansion. By Theorem 4.6, deciding whether A has a consistent moderately grounded expansion is $\mathbf{\Sigma}_3^P$-hard, even if every moderately grounded expansion of the premise set is consistent. Thus $\mathbf{\Pi}_3^P$-hardness of the problem under the asserted restriction follows. □

5 Normal form

An interesting issue is the complexity of autoepistemic reasoning in the case where premise sets are in some normalized form. It is pointed out in [11, Proposition 3.9] that each set $T \subseteq \mathcal{L}_L$ has a **K45** equivalent set in which each sentence is of the form

$$L\varphi_1 \vee \cdots \vee L\varphi_m \vee \neg L\psi_1 \vee \cdots \vee \neg L\psi_n \vee \omega$$

where all φ_i, ψ_j and ω are objective formulae, and all disjuncts except ω may be absent; there exists such a finite set if T is finite. Actually, formulae where $n \leq 1$ suffice for this purpose. We refer in the sequel to premise sets in the more general form as *normalized* premise sets and to those in the more strict format ($n \leq 1$) as *K-normal* premise sets. All lower complexity bounds derived for normal form in this section carry over to K-normal form.

The considered normal form is more restricted than Moore's normal form [20, 16], according to which each $\varphi \in \mathcal{L}_L$ can be represented by an equivalent formula $\Theta_1 \wedge \cdots \wedge \Theta_k$, where

$$\Theta_i = L\varphi_{i,1} \vee \cdots \vee L\varphi_{i,m_i} \vee \neg L\psi_{i,1} \vee \cdots \vee \neg L\psi_{i,n_i} \vee \omega_i,$$

and $\omega_i \in \mathcal{L}$ for all $1 \leq i \leq k$.

In particular, the considered normal form does not allow nestings of L operators, which entails that the corresponding language constitutes a rather small fragment of the language $\mathcal{L}_L$. However, in the context of consistent stable sets and stable expansions, studies of autoepistemic logic can be simplified (without loss of generality) by restriction to premise sets from this fragment; replacing an arbitrary premise set A with a normalized premise set A' equivalent to A with respect to stable sets and stable expansions may result in a large (exponential) increase in the size of the premise set (cf. [16, Proposition 3.5,4.4] and pp. 601,602 *ibid*).

It turns out that under the considered format, reasoning with moderately grounded expansions is most probably easier than in the general case and not harder than in standard AEL. This is in contrast to standard AEL expansions and parsimonious stable expansions, for which reasoning from normalized premise sets has the same complexity as in the general case.

Let us first consider standard AEL expansions. The following theorem strengthens Proposition 2.5.

Theorem 5.1 *Let A be a premise set in normalized form. Then, (i) deciding whether A has any stable expansion is $\mathbf{\Sigma}_2^P$-complete; (ii) deciding whether $\varphi \in \mathcal{L}_L$ occurs in some stable expansion of A is $\mathbf{\Sigma}_2^P$-complete; and (iii) deciding whether $\varphi \in \mathcal{L}_L$ occurs in all stable expansions of A is $\mathbf{\Pi}_2^P$-complete. $\mathbf{\Sigma}_2^P$-hardness of (i) and (ii) and $\mathbf{\Pi}_2^P$-hardness of (iii) hold even if A is in K-normal form and every stable expansion of A is consistent.*

Proof. The membership parts are obvious by Proposition 2.5.

The hardness parts are shown by suitable transformations of deciding whether for a QBF $\Phi = \exists y_1 \cdots \exists y_m \forall z_1 \cdots \forall z_l F$ it holds that $\Phi \in \mathrm{QBF}_{2,\exists}$ resp. $\Phi \notin \mathrm{QBF}_{2,\exists}$. Construct the following normalized set of premises:

$$A = \{\neg Ly_1 \vee y_1,\ Ly_1 \vee \neg y_1, \ldots, \neg Ly_m \vee y_m,\ Ly_m \vee \neg y_m,\ LF \vee \bot\}$$

Note that $A \cup A^L$ is consistent, hence A has only consistent stable expansions. Now consider the three problems of deciding whether

(a) A has a stable expansion,

(b) $\top$ occurs in a stable expansion of A,

(c) $\bot$ occurs in all stable expansions of A.

Clearly, (a) and (b) are equivalent problems, and since A has only consistent stable expansions, (c) is a complementary problem to (a). It is easy to see that the premise set

$$A' = \{Ly_1 \leftrightarrow y_1, \ldots, Ly_m \leftrightarrow y_m, LF\}$$

is logically equivalent to A. In [9, Proof of Theorem 4.1] it is shown that A' has a stable expansion iff Φ is valid. It follows that (a) and (b) are $\mathbf{\Sigma}_2^P$-hard and that (c) is $\mathbf{\Pi}_2^P$-hard. Since A is in K-normal form, the result follows. □

Next we consider parsimonious stable expansions, for which normal form of premise sets also does not affect the complexity of reasoning.

Theorem 5.2 *Let A be a premise set in normal form. Then, (i) deciding whether A has any parsimonious stable expansion is $\mathbf{\Sigma}_2^P$-complete; (ii) deciding whether $\varphi \in \mathcal{L}_L$ occurs in some parsimonious stable expansion of A is $\mathbf{\Sigma}_3^P$-complete; and (iii) deciding whether $\varphi \in \mathcal{L}_L$ occurs in all parsimonious stable expansions of A is $\mathbf{\Pi}_3^P$-complete. $\mathbf{\Sigma}_2^P$-hardness of (i), $\mathbf{\Sigma}_3^P$-hardness and (ii), and $\mathbf{\Pi}_3^P$-hardness of (iii) hold even if A is in K-normal form and every parsimonious stable expansion of A is consistent.*

Proof. $\mathbf{\Sigma}_2^P$-completeness of (i) follows immediately from arguments in the proof of Proposition 3.2 and Theorem 5.1. For (ii) and (iii), we observe that each formula in the premise set A constructed in the proof of Theorem 3.4 is equivalent to a small set of formulae in K-normal form. Each formula $Lx_i \leftrightarrow x_i$ is equivalent to $\{\neg Lx_i \vee x_i, Lx_i \vee \neg x_i\}$, $Ly_i \rightarrow y_i$ to $\{\neg Ly_i \vee y_i\}$, $L\neg y_i' \rightarrow \neg y_i'$ to $\{\neg L\neg y_i' \vee \neg y_i'\}$, and the formula $L(\bigwedge_{i=1}^m (y_i \leftrightarrow y_i') \rightarrow \neg F)$ to $\{L(\bigwedge_{i=1}^m (y_i \leftrightarrow y_i') \rightarrow \neg F) \vee \bot\}$. All other formulae in A are objective and thus already in K-normal form. Consequently, A can be replaced by an equivalent premise set in K-normal form in polynomial time. Thus by proofs analogous to those of Theorems 3.4,3.5, the theorems follows. $\Box$

Now let us turn to moderately grounded expansions. Normalized premise sets lower the complexity of reasoning by one level of the polynomial hierarchy, and locate the problems at the second level. More precisely, the problems are complete for the same complexity classes as the respective problems under standard AEL expansions. We start with the following lemma.

Lemma 5.3 *Given a set of premises A, deciding whether there exists a consistent stable set S such that $S \supseteq A$ is in* **NP**.

Proof. From Lemma 4.2 it is immediate that we may require without loss of generality that S is a consistent stable expansion of $A' = A \cup \{\varphi : L\varphi \in \Gamma\}$, where $\Gamma = S \cap A^{\pm L}$ is the kernel of S. Now proceed as follows. Guess $\Gamma \subseteq A^{\pm L}$ and truth assignments $\nu_0, \nu_1, \ldots, \nu_m$ to the atoms in A, where $m = |A^L|$. The guess is valid if $A' \cup \Gamma$ is satisfied by ν_0 and $A' \cup \Gamma \cup \{\neg\varphi_i\}$ is satisfied by ν_i for $1 \leq i \leq n$, where $\{\varphi_1, \ldots, \varphi_n\} = \{\varphi : \neg L\varphi \in \Gamma\}$. This holds because in this case Γ is A'-full and the corresponding stable expansion is consistent. The space needed to represent the guess Γ and $\nu_0, \ldots, \nu_m$ is clearly polynomial in the input size, and the guess can be verified in polynomial time. Thus the existence of a suitable S can be decided with an **NP** algorithm, and the lemma follows. $\Box$

Lemma 5.4 *Let A be a premise set in normal form, and let Λ be the kernel of a stable expansion E of A. Deciding whether E is moderately grounded is in Δ_2^P.*

Proof. Since A is normalized, each formula in A is of type

$$L\varphi_1 \vee \cdots \vee L\varphi_m \vee \neg L\psi_1 \vee \cdots \vee \neg L\psi_n \vee \omega.$$

Denote by Ω the set of the propositional parts ω of the formulae in A. It is not hard to see that $F_A(E)$ is logically equivalent to $\Omega \cap E$.

We construct a premise set A'' from A and Λ as follows:

$$A'' = A \cup \{\neg L\varphi : \neg L\varphi \in \Lambda\} \cup \{\neg L\omega : \omega \in \Omega - E\} \cup \{\neg L\omega_1 \vee \cdots \vee \neg L\omega_k \vee \bot\},$$

where $\{L\omega_1, \ldots, L\omega_k\} = \{L\omega : \omega \in \Omega \cap E\}$. For every $\omega \in \Omega$, deciding whether $\omega \in \Omega - E$ or $\omega \in \Omega \cap E$ is by Proposition 2.3 possible in polynomial time with an **NP** oracle. Consequently, A'' can be constructed in polynomial time with an **NP** oracle.

We claim that there exists a consistent stable set S such that $S \supseteq A''$ iff E is not moderately grounded.

Assume E is not moderately grounded. That is, there exists a stable set $S \supseteq A$ such that $S \prec E$. We notice that S is consistent. Consider $\neg L\psi \in \{\neg L\varphi : \neg L\varphi \in$

$\Lambda\} \cup \{\neg L\omega : \omega \in \Omega - E\}$. Such a $\neg L\psi$ exists only if E is consistent, and in this case it follows $\psi \notin E$. Since $S \prec E$ and $\psi \in \mathcal{L}$, it follows that $\psi \notin S$ and hence $\neg L\psi \in S$. Finally, consider the formula $\Theta = \neg L\omega_1 \vee \cdots \vee \neg L\omega_k \vee \bot$. We show that $\Theta \in S$ holds. Assume $\Theta \notin S$. Since Θ is a disjunction of negated modal atoms $\neg L\omega_i$ and $\bot$, it follows that $L\omega_i \in S$, for all $1 \leq i \leq k$. This implies $\{\omega_1, \ldots, \omega_k\} = \Omega \cap E \subseteq S$. Since $\Omega \cap E$ is logically equivalent to $F_A(E)$, it follows from Lemma 2.7 that $P(E) \subseteq P(S)$, i.e. $E \preceq S$. However, this is in contradiction to $E \not\preceq S$, which is implied by the assertion that $S \prec E$. Consequently, $\Theta \in S$ must hold. It follows $S \supseteq A''$. Thus the *if* direction is proved.

Conversely, assume there exists a consistent stable set $S \supseteq A''$. We may by Lemma 4.2 assume that S is a stable expansion E' of $A' = A'' \cup \{\varphi : L\varphi \in \Gamma\}$ where $\Gamma = E' \cap A''^{\pm L}$ is the kernel of E'. We observe that $A''^L = A^L \cup \{L\omega : \omega \in \Omega\}$ and that the set $\{\varphi : L\varphi \in \Gamma\}$ contains only objective formulae. Furthermore, for each $\omega \in \Omega$, if $\omega \in E'$, then $L\omega \in E'$ and hence $L\omega \in \Gamma$ holds. It is thus not hard to see from the structure of A' that $F_{A'}(E')$ is equivalent to the set $G = \{\varphi : L\varphi \in \Gamma\}$. Now consider $L\varphi \in \Gamma$. If $L\varphi \in A^L$, then $L\varphi \in \Lambda$ and hence $L\varphi \in E$, for otherwise $L\varphi, \neg L\varphi \in E'$ would hold, contradicting the consistency of E'; if $L\varphi \in \{L\omega : \omega \in \Omega\}$, we infer $L\varphi \in E$ by an analogous argument. Consequently, for each $L\varphi \in \Gamma$, it holds that $L\varphi \in E$, and thus $\varphi \in E$. It follows that $G \subseteq E$. Thus from Lemma 2.7, it follows that $P(E') \subseteq P(E)$, i.e. $E' \preceq E$. On the other hand, $E \preceq E'$ does not hold. Indeed, since $\neg L\omega_1 \vee \cdots \vee \neg L\omega_k \vee \bot \in E'$ and E' is consistent, it follows that $\neg L\omega_i \in E'$ for some i. This implies that $\omega_i \notin E'$. However, $\omega_i \in E$ holds from the construction of A''. Since ω_i is an objective formula, it follows $P(E) \not\subseteq P(E')$, i.e. $E \not\preceq E'$. Thus we have that $E' \preceq E$, $E \not\preceq E'$, i.e. $E' \prec E$. Since $E' \supseteq A$, it follows that E is not moderately grounded. Thus the *only if* direction holds, and the claim is proved.

By Lemma 5.3, deciding whether there exists for A'' a consistent stable set S such that $S \supseteq A''$ is possible with one call to an **NP** oracle. Thus given A and Λ, deciding whether E is moderately grounded is possible in polynomial time with an **NP** oracle, and the lemma follows. □

We thus obtain the following.

Theorem 5.5 *Let A be a premise set in normal form. Then, (i) deciding whether A has a moderately grounded expansion is $\mathbf{\Sigma}_2^P$-complete; (ii) deciding whether $\varphi \in \mathcal{L}_L$ occurs in some stable expansion of A is $\mathbf{\Sigma}_2^P$-complete; and (iii) deciding whether $\varphi \in \mathcal{L}_L$ occurs in all stable expansions of A is $\mathbf{\Pi}_2^P$-complete. $\mathbf{\Sigma}_2^P$-hardness of $(i), (ii)$ and $\mathbf{\Pi}_2^P$-hardness of (iii) hold even if A is in K-normal form and every stable expansion of A is consistent.*

Proof. The key for all membership proofs is that a guess $\Lambda \subseteq A^{\pm L}$ on the kernel of a moderately grounded expansion E of A can be verified in polynomial time with an **NP** oracle. This holds since deciding whether Λ is A-full is in Δ_2^P (Proposition 2.1) and deciding whether the stable expansion corresponding to Λ is moderately grounded is in Δ_2^P (Lemma 5.4). Consequently, (i) is clearly in $\mathbf{\Sigma}_2^P$.

On a successful guess Λ, deciding whether $\varphi \in E$ is in Δ_2^P (Corollary 2.4). Consequently, (ii) is in $\mathbf{\Sigma}_2^P$. Likewise deciding whether $\varphi \notin E$ is in Δ_2^P. This implies that deciding whether φ does not occur in some moderately grounded expansion of A is in $\mathbf{\Sigma}_2^P$. It follows from this that the complementary problem, i.e. (iii), is in $\mathbf{\Pi}_2^P$.

The hardness parts are shown by proving a property of the premise set

$$A = \{\neg Ly_1 \vee y_1,\ Ly_1 \vee \neg y_1, \ldots, \neg Ly_m \vee y_m,\ Ly_m \vee \neg y_m,\ LF \vee \bot\}$$

in the proof of Theorem 5.1. We show that each stable expansion E of A is moderately grounded. Hence, we may replace "stable expansion" in (a)-(c) in the proof of Theorem 5.1 equivalently with "moderately grounded expansion", and we immediately obtain the asserted hardness results.

Assume that E is not moderately grounded. Then, by Theorem 4.3, there exists $\Gamma \subseteq A^{\pm L}$ such that Γ is full for $A' = A \cup \{\varphi : L\varphi \in \Gamma\}$ and $E' \prec E$ where Γ is the kernel of the stable expansion E' of A'. Let $\Lambda = \Lambda_A(E)$. Clearly, $LF \in \Lambda$ and $LF \in \Gamma$. Since A has only consistent stable expansions, E, E' are consistent. Consequently, $Ly_i \in E$ iff $y_i \in E$ and $\neg Ly_i \in E$ iff $\neg y_i \in E$ holds, for all $1 \leq i \leq m$. Similarly, $Ly_i \in E'$ iff $y_i \in E'$ and $\neg Ly_i \in E'$ iff $\neg y_i \in E'$ holds, for all $1 \leq i \leq m$. Since $E' \prec E$ implies $P(E') \subseteq P(E)$, it follows $\Gamma = \Lambda$. Consequently, $A \cup \Lambda \models A' \cup \Gamma$ and $A' \cup \Gamma \models A \cup \Lambda$, and hence $F_A(E) \equiv F_{A'}(E')$. Thus by Lemma 2.7 $P(E) = P(E')$, which means $E' \not\prec E$, contradiction. Consequently, E is moderately grounded. $\Box$

6 Discussion and Conclusion

The complexity results in the previous sections show that reasoning with parsimonious stable expansions is most likely much harder than reasoning with all stable expansions, which is at the second level of the polynomial hierarchy PH. The same holds for reasoning with moderately grounded expansions. As a consequence, brave reasoning in these strengthened versions of standard AEL cannot be polynomially transformed into brave or cautious reasoning in standard AEL, unless $\mathbf{\Sigma}_3^P = \mathbf{\Sigma}_2^P$ or $\mathbf{\Sigma}_3^P = \mathbf{\Pi}_2^P$, which is considered very unlikely. For cautious reasoning, we have an analogous result.

In practical terms, this means that even if we have arbitrarily many oracle calls for brave or cautious reasoning in standard AEL for free, it is unlikely that we can compute the answer for brave reasoning in the strengthened versions in polynomial time. The same holds for cautious reasoning. Notice, however, that if φ is an objective formula, cautious reasoning with parsimonious stable expansions is no harder than cautious reasoning in standard AEL, and in fact is of the same difficulty.

The reason for the extreme intractability of parsimonious stable expansions and moderately grounded expansions are the following three sources of complexity:

1. logical consequence in classical propositional logic ($\models$)
2. the large (exponential) number of candidates E for stable expansions of A
3. for each stable expansion E of A, the (potentially exponential) number of candidates for a stable expansion E' of A (stable set $S \supseteq A$) such that $E' \prec E$ ($S \prec E$),

where "exponential" refers to the number of distinct modal atoms in the premise set A. Sources 1 and 2 are already present in Moore's formulation of AEL, while source 3 is introduced by parsimony and moderate groundedness.

To gain tractability, all three sources have to be eliminated. A straightforward but unsatisfactory way to achieve this is to bound the length of the encoding of A by a constant. More practical restrictions are yet to be found.

Our results on reasoning from premise sets where the knowledge is represented in a normalized format show an interesting effect of strengthening AEL on the complexity of autoepistemic reasoning. Under this format, brave and cautious reasoning are at the second level of the polynomial hierarchy if standard AEL expansions are considered. The problems are lifted to the third level if standard AEL expansions are restricted to parsimonious stable expansions. However, the further restriction from parsimonious stable expansions to moderately grounded expansions locates the reasoning tasks at the second level of the polynomial hierarchy again. Thus the complexity of autoepistemic reasoning from normalized premise sets is nonmonotonic in these successive strengthenings of AEL.

Reasoning with parsimonious or moderately grounded autoepistemic expansions is, to our best knowledge, the first problem in AI known to be $\mathbf{\Sigma}_3^P$-complete. Actually, few practical problems of this complexity are known to date (see [7] for others).

We note that more recently, higher-level complexity results within PH have been derived for other forms of non-classical reasoning. (See also [1] for a comprehensive survey of the field.)

Nonmonotonic Logics. Gottlob [9] has shown that several reasoning tasks in a number of nonmonotonic propositional logics are complete for some classes of the second level of PH. In particular, he showed that besides autoepistemic logic, in Reiter's default logic [26], in McDermott and Doyle's nonmonotonic logic [17, 18], and in Marek and Truszczyński's nonmonotonic logic N [15], which all have a fixed point semantics, deciding whether a fixed point exists is $\mathbf{\Sigma}_2^P$-complete, deciding whether a formula belongs to some fixed point is $\mathbf{\Sigma}_2^P$-complete, and deciding whether a formula belongs to all fixed points is $\mathbf{\Pi}_2^P$-complete. For default logic, similar results have been independently obtained by Stillman [30] and Papadimitriou and Sideri [25].

Revision and Update of Propositional Theories. Several operators $\circ$ for revising or updating a knowledge base (theory) T with a sentence F have been proposed which handle arising inconsistencies appropriately. Nebel [22] and Eiter and Gottlob [6] have shown that for almost all update operators $\circ$, deciding whether the revised knowledge base $T \circ F$ implies a formula G is at the second level of PH and for many of them $\mathbf{\Pi}_2^P$-complete.

Closed World Reasoning and Circumscription. Eiter and Gottlob [5] have shown that inference with a propositional theory under various forms of the closed world assumption and under circumscription is at the second level of PH. In particular, deciding whether the circumscription $CIRC(F)$ logically implies a formula G, i.e. G is satisfied in every minimal model of F, is shown to be $\mathbf{\Pi}_2^P$-complete.

TMS. Rutenburg [27] has shown that for a certain variant of truth maintenance system (TMS) [4] deciding whether there exists a "nogood" of certain size is $\mathbf{\Sigma}_2^P$-complete.

Abduction. Eiter and Gottlob [7] analyzed the complexity of logical abduction [3] in the full propositional context. It appears that deciding whether an abduction problem has a solution is $\mathbf{\Sigma}_2^P$-complete. The same holds for checking a certain property of hypotheses, i.e. abducible propositions. Moreover, we show that special variants of abduction are even $\mathbf{\Sigma}_3^P$-complete.

All these results suggest that nonmonotonic reasoning is more complex than classical reasoning in the propositional context; for logic programs and the first-order case, this is definitively known for many approaches, cf. [26, 28, 2, 13].

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